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editorial

Growth of Mathematical Concepts

The history of the growth of Mathematics is as interesting as a good detective novel. Arithmetic has a long history starting with addition, subtraction, multiplication and division, most of the work in commerce could be managed.

In Indian system, astronomers were able to calculate many calculations such as the prediction of the motion of the planets, solar and lunar eclipses, the position of the stars and so on very accurately with the help of conic shells. Modern prediction of the time of eclipses coincide with Indian calculation.

For fast multiplication and division, logarithm was discovered starting with the normal four digit or decimal places, one could even go to 8 decimal logarithm tables. Ever since the discovery of the powerful computers, advances in Physics, Chemistry and Biology went hand in hand.

The application of Mathematics to Physics, crystallography, spectroscopy and diffraction gave a big boost for the advance of Physics, Chemistry and Biology, particularly for the study of the structure of molecules and crystals. Will 1 + 1 give 2 always? In the study of electrical circuits, this may not be true. There are a large number of great mathematicians. The great Ramanujam made his mark in Cambridge. Einstein effaced the borders of Mathematics and Physics. Normally one makes a premise and develops theories based on experimental facts. In the case of Einstein, first he made conclusions and they were proved later by experiments. To study higher Mathematics gives one great joy and satisfaction.

We wish you all the best for your future courses of study.

Anil Ahlawat
Editor

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.
During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.
Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 151

JEE MAIN

1. If $r, s \in \{1, 2, 3, ..., 20\}$, then the sum of the digits of $S = \sum_{r<s} (r-s)^2$ is

(a) 7 (b) 9 (c) 10 (d) 11

2. If $I = \int_0^1 (1-x^2)^9 x^9 \, dx$, then the sum of the digits of $\frac{1}{I}$ is

(a) 2 (b) 4 (c) 8 (d) 9

3. Three normals are drawn from the point (20, 8) to the parabola $y^2 = 4x$. The radius of the circle through the feet of the normals is

(a) $\sqrt{5}$ (b) 5 (c) $5\sqrt{5}$ (d) 10

4. From a point P tangents are drawn to the circle $x^2 + y^2 = a^2$. If the chord of contact touches the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the locus of P is

(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2}{b^2}$ (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{b^2}{a^2}$

(c) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{a^2}{b^2}$ (d) $\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{b^2}{a^2}$

5. If events A and B are such that $P(\bar{A}) = 0.3$, $P(B) = 0.4$ and $P(A \cap \bar{B}) = 0.5$, then $P(B/(A \cup \bar{B})) =$

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{2}{5}$ (d) $\frac{1}{4}$

JEE ADVANCED

6. The sides of an acute angled triangle are 10, 24, x. The integer values of x are

(a) 22 (b) 23 (c) 24 (d) 25

COMPREHENSION

The vertices of a tetrahedron are $O(0, 0, 0)$, $A(1, 2, 3)$, $B(2, 3, 1)$, $C(3, 1, 2)$.

7. The distance of the vertex A from the edge BC is

(a) $\frac{5\sqrt{3}}{6}$ (b) $\frac{6\sqrt{3}}{5}$ (c) $\frac{3}{\sqrt{2}}$ (d) $\sqrt{\frac{3}{2}}$

8. The distance of the vertex A from the face OBC is

(a) $\frac{5\sqrt{3}}{6}$ (b) $\frac{6\sqrt{3}}{5}$ (c) $3\sqrt{2}$ (d) $2\sqrt{3}$

INTEGER MATCH

9. Let $C_r = \binom{10}{r}$ and $S = \frac{C_0}{5} - \frac{C_1}{6} + \frac{C_2}{7} - \frac{C_3}{8} + ... \ ... + \frac{C_{10}}{15}$. The number of prime factors of $\frac{1}{S}$ is

MATCHING LIST

10. The letters of the word CALCULUS are used to form 3, 4, 5, 6-letter words.

Column-I		Column-II	
P.	The sum of the digits of the number of 3-letter words is	1.	3
Q.	The sum of the digits of the number of 4-letter words is	2.	12
R.	The sum of the digits of the number of 5-letter words is	3.	15
S.	The sum of the digits of the number of 6-letter words is	4.	18

	P	Q	R	S
(a)	3	4	1	2
(b)	4	1	2	3
(c)	3	2	1	4
(d)	2	3	4	1

See Solution set of Maths Musing 150 on page no. 30

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

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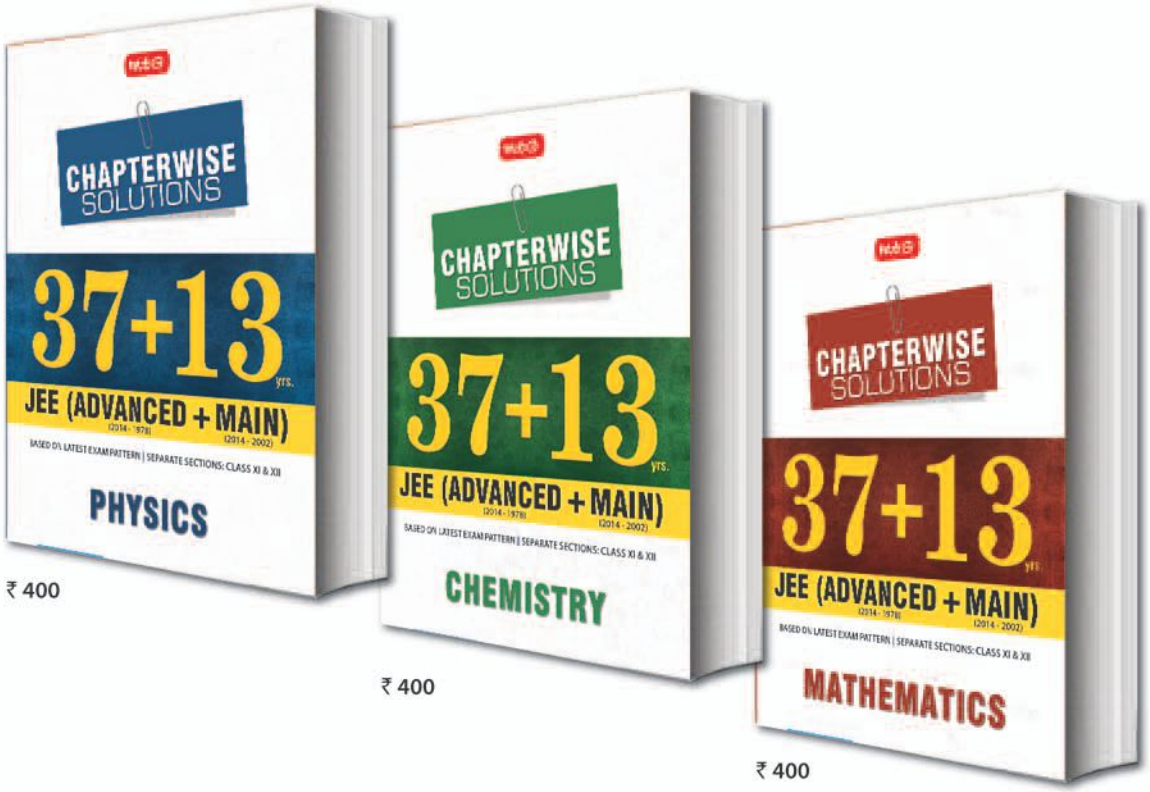
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JEE Advanced

SOLVED PAPER 2015

*ALOK KUMAR, B.Tech, IIT Kanpur

PAPER-1

SECTION-1 (Maximum Marks : 32)

This section contains EIGHT questions. The answer to each question is a SINGLE DIGIT INTEGER ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme: +4 If the bubble corresponding to the answer is darkened, 0 In all other cases.

1. The number of distinct solutions of the equation

$$\frac{5}{4}\cos^2 2x + \cos^4 x + \sin^4 x + \cos^6 x + \sin^6 x = 2$$

in the interval $[0, 2\pi]$ is

2. Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is

3. The minimum number of times a fair coin needs to be tossed, so that the probability of getting at least two heads is at least 0.96, is

4. Let n be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that all the girls stand consecutively in the queue. Let m be the number of ways in which 5 boys and 5 girls can stand in a queue in such a way that exactly four girls stand consecutively in the queue. Then the value of $\frac{m}{n}$ is

5. If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is

6. Let $f : R \to R$ be a function defined by $f(x) = \begin{cases} [x], & x \le 2 \\ 0, & x > 2 \end{cases}$, where $[x]$ is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{xf(x^2)}{2 + f(x+1)}\,dx$, then the value of $(4I - 1)$ is

7. A cylindrical container is to be made from certain solid material with the following constraints : It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.
If the volume of the material used to make the container is minimum when the inner radius of the container is 10 mm, then the value of $\frac{V}{250\pi}$ is

8. Let $F(x) = \int_{x}^{x^2 + \frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in R$ and $f : \left[0, \frac{1}{2}\right] \to [0, \infty)$ be continuous function. For $a \in \left[0, \frac{1}{2}\right]$, if $F'(a) + 2$ is the area of the region bounded by $x = 0$, $y = 0$, $y = f(x)$ and $x = a$, then $f(0)$ is

SECTION-2 (Maximum Marks : 40)

This section contains TEN questions. Each question has FOUR options (a), (b), (c) and (d). ONE OR MORE THAN ONE of these four option(s) is (are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 If only the bubble(s) corresponding to all the correct option(s) is (are) darkened, 0 If none of the bubbles is darkened, –2 In all other cases.

9. Let X and Y be two arbitrary, 3×3 non-zero, skew-symmetric matrices and Z be an arbitrary 3×3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric?

(a) $Y^3Z^4 - Z^4Y^3$ (b) $X^{44} + Y^{44}$
(c) $X^4Z^3 - Z^3X^4$ (d) $X^{23} + Y^{23}$

*Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

10. Which of the following values of α satisfy the equation

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha ?$$

(a) -4 (b) 9 (c) -9 (d) 4

11. In R^3, consider the planes $P_1 : y = 0$ and $P_2 : x + z = 1$. Let P_3 be a plane, different from P_1 and P_2, which passes through the intersection of P_1 and P_2. If the distance of the point $(0, 1, 0)$ from P_3 is 1 and the distance of a point (α, β, γ) from P_3 is 2, then which of the following relations is (are) true?

(a) $2\alpha + \beta + 2\gamma + 2 = 0$
(b) $2\alpha - \beta + 2\gamma + 4 = 0$
(c) $2\alpha + \beta - 2\gamma - 10 = 0$
(d) $2\alpha - \beta + 2\gamma - 8 = 0$

12. In R^3, let L be a straight line passing through the origin. Suppose that all the points on L are at a constant distance from the two planes $P_1 : x + 2y - z + 1 = 0$ and $P_2 : 2x - y + z - 1 = 0$. Let M be the locus of the feet of the perpendiculars drawn from the points on L to the plane P_1. Which of the following points lie(s) on M?

(a) $\left(0, -\frac{5}{6}, -\frac{2}{3}\right)$ (b) $\left(-\frac{1}{6}, -\frac{1}{3}, \frac{1}{6}\right)$

(c) $\left(-\frac{5}{6}, 0, \frac{1}{6}\right)$ (d) $\left(-\frac{1}{3}, 0, \frac{2}{3}\right)$

13. Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle OPQ is $3\sqrt{2}$, then which of the following is (are) the coordinates of P?

(a) $(4, 2\sqrt{2})$ (b) $(9, 3\sqrt{2})$

(c) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (d) $(1, \sqrt{2})$

14. Let $y(x)$ be a solution of the differential equation $(1 + e^x)y' + ye^x = 1$. If $y(0) = 2$, then which of the following statements is (are) true?

(a) $y(-4) = 0$
(b) $y(-2) = 0$
(c) $y(x)$ has a critical point in the interval $(-1, 0)$
(d) $y(x)$ has no critical point in the interval $(-1, 0)$

15. Consider the family of all circles whose centers lie on the straight line $y = x$. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P, Q are functions of x, y and y' (here $y' = \frac{dy}{dx}$, $y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true?

(a) $P = y + x$
(b) $P = y - x$
(c) $P + Q = 1 - x + y + y' + (y')^2$
(d) $P - Q = x + y - y' - (y')^2$

16. Let $g : R \to R$ be a differentiable function with $g(0) = 0$, $g'(0) = 0$ and $g'(1) \neq 0$. Let

$$f(x) = \begin{cases} \frac{x}{|x|} g(x), & x \neq 0 \\ 0, & x = 0 \end{cases} \text{ and } h(x) = e^{|x|} \text{ for all } x \in R.$$

Let $(foh)(x)$ denote $f(h(x))$ and $(hof)(x)$ denote $h(f(x))$. Then which of the following is (are) true?

(a) f is differentiable at $x = 0$
(b) h is differentiable at $x = 0$
(c) foh is differentiable at $x = 0$
(d) hof is differentiable at $x = 0$

17. Let $f(x) = \sin\left(\frac{\pi}{6}\sin\left(\frac{\pi}{2}\sin x\right)\right)$ for all $x \in R$ and $g(x) = \frac{\pi}{2}\sin x$ for all $x \in R$. Let $(fog)(x)$ denote $f(g(x))$ and $(gof)(x)$ denote $g(f(x))$. Then which of the following is (are) true?

(a) Range of f is $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(b) Range of fog is $\left[-\frac{1}{2}, \frac{1}{2}\right]$
(c) $\lim_{x \to 0} \frac{f(x)}{g(x)} = \frac{\pi}{6}$
(d) There is an $x \in R$ such that $(gof)(x) = 1$

18. Let ΔPQR be a triangle. Let $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{RP}$ and $\vec{c} = \overrightarrow{PQ}$. If $|\vec{a}| = 12$, $|\vec{b}| = 4\sqrt{3}$ and $\vec{b} \cdot \vec{c} = 24$, then which of the following is (are) true?

(a) $\frac{|\vec{c}|^2}{2} - |\vec{a}| = 12$
(b) $\frac{|\vec{c}|^2}{2} + |\vec{a}| = 30$
(c) $|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = 48\sqrt{3}$
(d) $\vec{a} \cdot \vec{b} = -72$

SECTION-3 (Maximum Marks : 16)

This section contains TWO questions. Each question contains two columns, Column I and Column II. Column I has four entries (A), (B), (C) and (D). Column II has five entries (P), (Q), (R), (S) and (T). Match the entries in Column I with the entries in Column II. One or more entries in Column I may match with one or more entries in Column II.

The ORS contains a 4 × 5 matrix whose layout will be similar to the one shown below:

(A) P Q R S T
(B) P Q R S T
(C) P Q R S T
(D) P Q R S T

For each entry in Column I, darken the bubbles of all the matching entries. For example, if entry (A) in Column I matches with entries (Q), (R) and (T), then darken these three bubbles in the ORS. Similarly, for entries (B), (C) and (D). Marking scheme : For each entry in column I, +2 If only the bubble(s) corresponding to all the correct match(es) is(are) darkened, 0 If none of the bubbles is darkened, –1 In all other cases

19.

	Column I	Column II
(A)	In R^2, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha=2+\sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P) 1
(B)	Let a and b be real numbers such that the function $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\geq 1\end{cases}$ is differentiable for all $x\in R$. Then possible value(s) of a is (are)	(Q) 2
(C)	Let $\omega\neq 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3}+(2+3\omega-3\omega^2)^{4n+3}+(-3+2\omega+3\omega^2)^{4n+3}=0$, then possible value(s) of n is (are)	(R) 3
(D)	Let the harmonic mean of two positive real numbers a and b be 4. If q is a positive real number such that $a, 5, q, b$ is in arithmetic progression, then the value(s) of $\lvert q-a\rvert$ is (are)	(S) 4
		(T) 5

20.

	Column I	Column II
(A)	In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $2(a^2-b^2)=c^2$ and $\lambda=\frac{\sin(X-Y)}{\sin Z}$, then possible values of n for which $\cos(n\pi\lambda)=0$ is (are)	(P) 1
(B)	In a triangle ΔXYZ, let a, b and c be the lengths of the sides opposite to the angles X, Y and Z, respectively. If $1+\cos 2X-2\cos 2Y=2\sin X\sin Y$, then possible value(s) of $\frac{a}{b}$ is (are)	(Q) 2
(C)	In R^2, let $\sqrt{3}\hat{i}+\hat{j}$, $\hat{i}+\sqrt{3}\hat{j}$ and $\beta\hat{i}+(1-\beta)\hat{j}$ be the position vectors of X, Y and Z with respect to the origin O, respectively. If the distance of Z from the bisector of the acute angle of $\overrightarrow{OX}$ with $\overrightarrow{OY}$ is $\frac{3}{\sqrt{2}}$, then possible value(s) of $\lvert\beta\rvert$ is (are)	(R) 3
(D)	Suppose that $F(\alpha)$ denotes the area of the region bounded by $x=0$, $x=2$, $y^2=4x$ and $y=\lvert\alpha x-1\rvert+\lvert\alpha x-2\rvert+\alpha x$, where $\alpha\in\{0,1\}$. Then the value(s) of $F(\alpha)+\frac{8}{3}\sqrt{2}$, when $\alpha=0$ and $\alpha=1$, is (are)	(S) 5
		(T) 6

PAPER-2

SECTION-1 (Maximum Marks : 32)

This section contains eight questions. The answer to each question is a single digit integer ranging from 0 to 9, both inclusive. For each question, darken the bubble corresponding to the correct integer in the ORS. Marking scheme : +4 if the bubble corresponding to the answer is darkened. 0 in all other cases.

1. For any integer k, let $\alpha_k=\cos\left(\frac{k\pi}{7}\right)+i\sin\left(\frac{k\pi}{7}\right)$ where $i=\sqrt{-1}$. The value of the expression

$$\frac{\sum_{k=1}^{12}\lvert\alpha_{k+1}-\alpha_k\rvert}{\sum_{k=1}^{3}\lvert\alpha_{4k-1}-\alpha_{4k-2}\rvert}$$ is

2. Suppose that all the terms of an arithmetic progression (A.P.) are natural numbers. If the ratio of the sum of the first seven terms to the sum of the first eleven terms is 6 : 11 and the seventh term lies in between 130 and 140, then the common difference of this A.P. is

3. The coefficient of x^9 in the expansion of $(1+x)(1+x^2)(1+x^3)\ldots(1+x^{100})$ is

4. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1, 0)$ and $(f_2, 0)$ where $f_1>0$ and $f_2<0$. Let P_1 and P_2 be two parabolas with a common vertex at (0, 0) and with foci at $(f_1, 0)$ and $(2f_2, 0)$ respectively. Let T_1 be a

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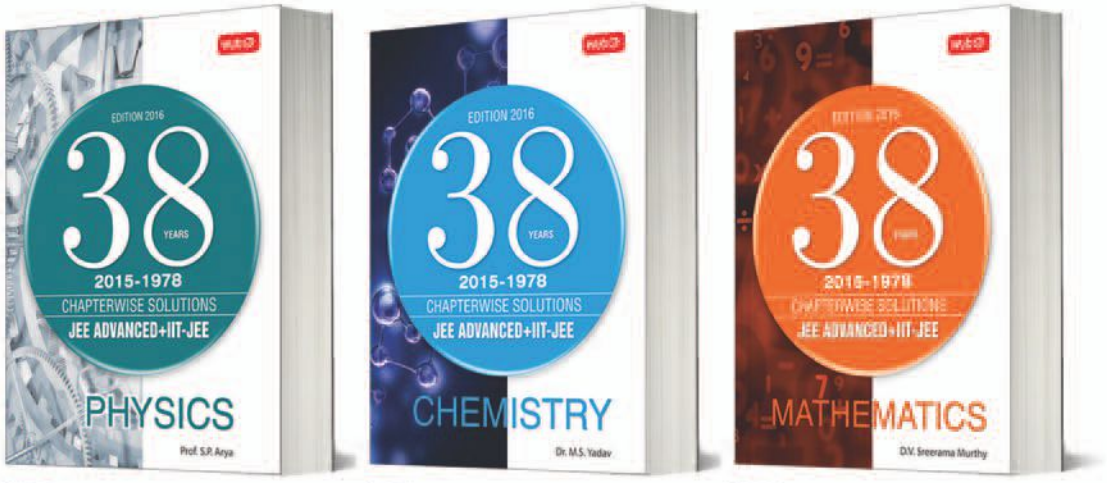

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tangent to P_1 which passes through $(2f_2, 0)$ and T_2 be a tangent to P_2 which passes through $(f_1, 0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is

5. Let m and n be two positive integers greater than 1. If $\lim\limits_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)}-e}{\alpha^m}\right)=-\left(\frac{e}{2}\right)$ then the value of $\frac{m}{n}$ is

6. If $\alpha=\int\limits_0^1 (e^{9x+3\tan^{-1}x})\left(\frac{12+9x^2}{1+x^2}\right)dx$ where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha|-\frac{3\pi}{4}\right)$ is

7. Let $f : R \to R$ be a continuous odd function, which vanishes exactly at one point and $f(1)=\frac{1}{2}$. Suppose that $F(x)=\int\limits_{-1}^{x} f(t)dt$ for all $x \in [-1, 2]$ and $G(x)=\int\limits_{-1}^{x} t|f(f(t))|dt$ for all $x \in [-1, 2]$. If $\lim\limits_{x\to 1}\frac{F(x)}{G(x)}=\frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is

8. Suppose that $\vec{p}, \vec{q}$ and $\vec{r}$ are three non-coplanar vectors in R^3. Let the components of a vector $\vec{s}$ along $\vec{p}, \vec{q}$ and $\vec{r}$ be 4, 3 and 5 respectively. If the components of this vector $\vec{s}$ along $(-\vec{p}+\vec{q}+\vec{r})$, $(\vec{p}-\vec{q}+\vec{r})$ and $(-\vec{p}-\vec{q}+\vec{r})$ are x, y and z respectively, then the value of $2x + y + z$ is

SECTION-2 (Maximum Marks : 32)

This section contains eight questions. Each question has four options (a), (b), (c) and (d). One or more than one of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 if only the bubble(s) corresponding to all the correct option(s) is(are) darkened. 0 if none of the bubbles is darkened. –2 in all other cases.

9. Let S be the set of all non-zero real numbers α such that the quadratic equation $\alpha x^2 - x + \alpha = 0$ has two distinct real roots x_1 and x_2 satisfying the inequality $|x_1 - x_2| < 1$. Which of the following intervals is(are) a subset(s) of S?

(a) $\left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right)$ (b) $\left(-\frac{1}{\sqrt{5}}, 0\right)$

(c) $\left(0, \frac{1}{\sqrt{5}}\right)$ (d) $\left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$

10. If $\alpha=3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta=3\cos^{-1}\left(\frac{4}{9}\right)$, where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are)

(a) $\cos\beta > 0$ (b) $\sin\beta < 0$

(c) $\cos(\alpha+\beta) > 0$ (d) $\cos\alpha < 0$

11. Let E_1 and E_2 be two ellipses whose centres are at the origin. The major axes of E_1 and E_2 lie along the x-axis and the y-axis respectively. Let S be the circle $x^2 + (y - 1)^2 = 2$. The straight line $x + y = 3$ touches the curves S, E_1 and E_2 at P, Q and R, respectively. Suppose that $PQ=PR=\frac{2\sqrt{2}}{3}$. If e_1 and e_2 are the eccentricities of E_1 and E_2 respectively, then the correct expression(s) is(are)

(a) $e_1^2+e_2^2=\frac{43}{40}$ (b) $e_1e_2=\frac{\sqrt{7}}{2\sqrt{10}}$

(c) $|e_1^2-e_2^2|=\frac{5}{8}$ (d) $e_1e_2=\frac{\sqrt{3}}{4}$

12. Consider the hyperbola $H : x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle PMN, then the correct expression(s) is(are)

(a) $\frac{dl}{dx_1}=1-\frac{1}{3x_1^2}$ for $x_1>1$

(b) $\frac{dm}{dx_1}=\frac{x_1}{3(\sqrt{x_1^2-1})}$ for $x_1>1$

(c) $\frac{dl}{dx_1}=1+\frac{1}{3x_1^2}$ for $x_1>1$

(d) $\frac{dm}{dy_1}=\frac{1}{3}$ for $y_1>0$

13. The option(s) with the values of a and L that satisfy the following equation is(are)

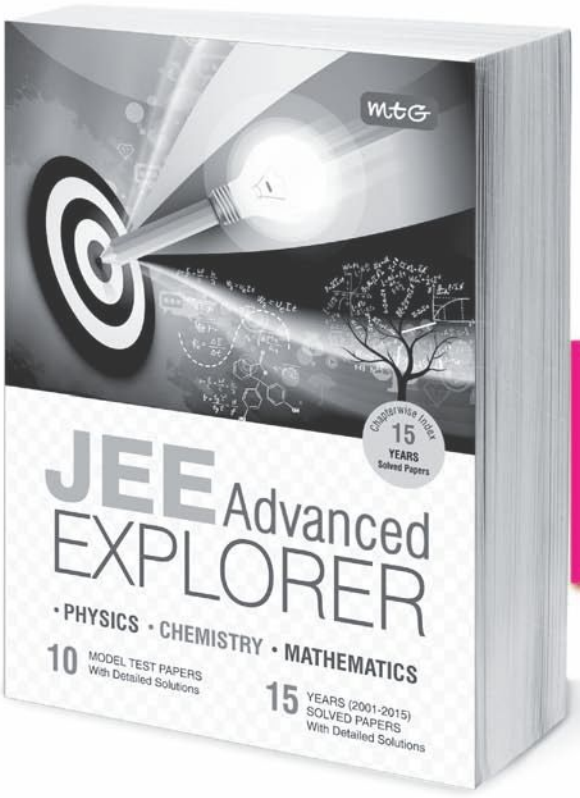
$$\frac{\int\limits_0^{4\pi} e^t(\sin^6 at+\cos^4 at)dt}{\int\limits_0^{\pi} e^t(\sin^6 at+\cos^4 at)dt}=L?$$

(a) $a=2, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$ (b) $a=2, L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

(c) $a=4, L=\frac{e^{4\pi}-1}{e^{\pi}-1}$ (d) $a=4, L=\frac{e^{4\pi}+1}{e^{\pi}+1}$

JEE (ADVANCED)

Dry runs are here!



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14. Let $f, g : [-1, 2] \to R$ be continuous functions which are twice differentiable on the interval $(-1, 2)$. Let the values of f and g at the points -1, 0 and 2 be as given in the following table.

	$x=-1$	$x=0$	$x=2$
$f(x)$	3	6	0
$g(x)$	0	1	-1

In each of the intervals $(-1, 0)$ and $(0, 2)$, the function $(f - 3g)''$ never vanishes. Then the correct statement(s) is(are)

(a) $f'(x) - 3g'(x) = 0$ has exactly three solutions in $(-1, 0) \cup (0, 2)$

(b) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(-1, 0)$

(c) $f'(x) - 3g'(x) = 0$ has exactly one solution in $(0, 2)$

(d) $f'(x) - 3g'(x) = 0$ has exactly two solutions in $(-1, 0)$ and exactly two solutions in $(0, 2)$

15. Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are)

(a) $\int_0^{\pi/4} xf(x)dx = \frac{1}{12}$ (b) $\int_0^{\pi/4} f(x)dx = 0$

(c) $\int_0^{\pi/4} x f(x)dx = \frac{1}{6}$ (d) $\int_0^{\pi/4} f(x)dx = 1$

16. Let $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$ for all $x \in R$ with $f\left(\frac{1}{2}\right) = 0$.

If $m \le \int_{1/2}^{1} f(x)dx \le M$, then the possible values of m and M are

(a) $m = 13, M = 24$ (b) $m = \frac{1}{4}, M = \frac{1}{2}$

(c) $m = -11, M = 0$ (d) $m = 1, M = 12$

SECTION-3 (Maximum Marks : 16)

This section contains two paragraphs. Based on each paragraph, there will be two questions. Each question has four options (a), (b), (c) and (d). One or more than one of these four option(s) is(are) correct. For each question, darken the bubble(s) corresponding to all the correct option(s) in the ORS. Marking scheme : +4 if only the bubble(s)corresponding to all the correct option(s) is(are) darkened. 0 if none of the bubbles is darkened. –2 in all other cases.

Paragraph-1

Let n_1 and n_2 be the number of red and black balls, respectively, in box I. Let n_3 and n_4 be the number of red and black balls, respectively, in box II.

17. One of the two boxes, box I and box II, was selected at random and a ball was drawn randomly out of this box. The ball was found to be red. If the probability that this red ball was drawn from box II is 1/3, then the correct option(s) with the possible values of n_1, n_2, n_3 and n_4 is(are)

(a) $n_1 = 3, n_2 = 3, n_3 = 5, n_4 = 15$

(b) $n_1 = 3, n_2 = 6, n_3 = 10, n_4 = 50$

(c) $n_1 = 8, n_2 = 6, n_3 = 5, n_4 = 20$

(d) $n_1 = 6, n_2 = 12, n_3 = 5, n_4 = 20$

18. A ball is drawn at random from box I and transferred to box II. If the probability of drawing a red ball from box I, after this transfer, is 1/3, then the correct option(s) with the possible values of n_1 and n_2 is (are)

(a) $n_1 = 4$ and $n_2 = 6$ (b) $n_1 = 2$ and $n_2 = 3$

(c) $n_1 = 10$ and $n_2 = 20$ (d) $n_1 = 3$ and $n_2 = 6$

Paragraph-2

Let $F : R \to R$ be a thrice differentiable function. Suppose that $F(1) = 0$, $F(3) = -4$ and $F'(x) < 0$ for all $x \in (1/2, 3)$. Let $f(x) = xF(x)$ for all $x \in R$.

19. The correct statement(s) is(are)

(a) $f'(1) < 0$ (b) $f(2) < 0$

(c) $f'(x) \neq 0$ for any $x \in (1, 3)$

(d) $f'(x) = 0$ for some $x \in (1, 3)$

20. If $\int_1^3 x^2 F'(x)dx = -12$ and $\int_1^3 x^3 F''(x)dx = 40$, then the correct expression(s) is(are)

(a) $9f'(3) + f'(1) - 32 = 0$

(b) $\int_1^3 f(x)dx = 12$

(c) $9f'(3) - f'(1) + 32 = 0$

(d) $\int_1^3 f(x)dx = -12$

SOLUTIONS

PAPER-1

1. **(8) :** The equation simplifies to

$$\frac{5}{4}\cos^2 2x + 1 - \frac{1}{2}\sin^2 2x + 1 - \frac{3}{4}\sin^2 2x = 2$$

$$\Rightarrow \frac{5}{4}(\cos^2 2x - \sin^2 2x) = 0$$

$\Rightarrow \cos 4x = 0$

As $x \in [0, 2\pi]$, we have $4x \in [0, 8\pi]$

So the number of values that $4x$ takes is 8 and as there is a one-one correspondence between the values of $4x$ and those of x, we have, the number of solutions as 8.

[Rating : Easy]

2. (4) : Let $P(t^2, 2t)$ be a point on $y^2 = 4x$ and its reflection be $P'(\alpha, \beta)$ which are given by

$$\frac{\alpha - t^2}{1} = \frac{\beta - 2t}{1} = -\frac{2(t^2 + 2t + 4)}{2}$$

giving $\alpha = -(2t + 4)$ and $\beta = -(t^2 + 4) = -5$

Thus $t = \pm 1$

So, the points are $A(-2, -5)$ and $B(-6, -5)$

$\therefore \quad AB = 4$

[Rating : Doable]

3. (8) : Let the coin be tossed n times

P(atleast two heads)

$$= 1 - \left(\frac{1}{2}\right)^n - {}^nC_1\left(\frac{1}{2}\right)^n \geq 0.96 \quad \text{(given)}$$

$$\Rightarrow \quad \frac{4}{100} \geq \frac{n+1}{2^n} \quad \Rightarrow \quad \frac{2^n}{n+1} \geq 25$$

The least value of 'n' is then 8 as can be found by checking.

[Rating : Difficult]

4. (5) : $m = \lfloor 5 \times {}^6C_2 \cdot {}^5C_4 \lfloor 2 \cdot \lfloor 4$ and $n = \lfloor 5 \times \lfloor 6$

Then $\frac{m}{n} = \frac{15 \times 2 \times \lfloor 5}{\lfloor 6} = 5$

[Rating : Doable]

5. (2) : The equation of normal at the end point of latus rectum *i.e.* at $(1, 2)$ is $y = -x + 3$

As $x + y - 3 = 0$ touches the circle $(x - 3)^2 + (y + 2)^2 = r^2$, we have

$$\left|\frac{3 - 2 - 3}{\sqrt{2}}\right| = r \quad \therefore \; r = \sqrt{2} \Rightarrow r^2 = 2$$

[Rating : Easy]

6. (0) : $I = \int_{-1}^{2} \frac{x[x^2]\,dx}{2 + [x + 1]}$

Splitting the integrand, we get

$$= \int_{-1}^{0} \frac{x \cdot 0}{2 + 0}dx + \int_{0}^{1} \frac{x \cdot 0}{2 + 1}dx + \int_{1}^{\sqrt{2}} \frac{x \cdot 1}{2 + 0}dx + 0$$

$$= \frac{1}{2}\int_{1}^{\sqrt{2}} x\,dx \;\; = \frac{x^2}{4}\Big|_1^{\sqrt{2}} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

$\therefore \quad 4I = 1 \quad \Rightarrow \quad 4I - 1 = 0$

[Rating : Doable]

7. (4) : Volume of material, $V = \pi r^2 h$

Total volume $V_1 = \pi(r + 2)^2 \cdot 2 + \pi\{(r + 2)^2 - r^2\} \cdot \frac{V}{\pi r^2}$

$$= V\frac{(r + 2)^2}{r^2} - V + 2\pi(r + 2)^2$$

Differentiating w.r.t. 'r', we get

$$\frac{dV_1}{dr} = 2V\left(\frac{r + 2}{r}\right) \cdot \left(\frac{-2}{r^2}\right) + 4\pi(r + 2)$$

$$= (r + 2) \times 4\left\{\pi - \frac{V}{r^3}\right\}$$

Put $\frac{dV_1}{dr} = 0$

$\therefore \quad V = \pi r^3 \qquad (\because \; r = 10 \text{ mm})$

$\Rightarrow \quad \frac{V}{\pi} = r^3 \quad \Rightarrow \quad \frac{V}{\pi} = 1000 \quad \Rightarrow \quad \frac{V}{250\pi} = 4$

[Rating : Doable]

8. (3) : We have, $F(x) = \int_{x}^{x^2 + \frac{\pi}{6}} 2\cos^2 t\,dt$

$F'(x) = 2\cos^2(x^2 + (\pi/6)) \cdot 2x - 2\cos^2 x$

(By Leibnitz formula)

We have, $F'(a) + 2 = \int_0^a f(x)\,dx$

Differentiating w.r.t. 'a', we get

$F''(a) = f(a)$

$f(0) = F''(0)$

Again differentiating $F'(x)$ w.r.t. 'x', we have

$F''(x) = 2x\{1 + \cos(2x^2 + \pi/3)\}' + 2\{1 + \cos(2x^2 + \pi/3)\} + 2\sin 2x$

Set $x = 0$

$$F''(0) = 2\left\{1 + \cos\frac{\pi}{3}\right\} + 0 = 2\left(1 + \frac{1}{2}\right) = 2 \cdot \frac{3}{2} = 3$$

[Rating : Difficult]

9. (c, d) : $X^T = -X$, $Y^T = -Y$, $Z^T = Z$

$(Y^3Z^4 - Z^4Y^3)^T = (Y^3Z^4)^T - (Z^4Y^3)^T$

$= -Z^4Y^3 + Y^3Z^4 = Y^3Z^4 - Z^4Y^3$

Hence, symmetric.

Similarly $(X^{44} + Y^{44})^T = X^{44} + Y^{44}$, so symmetric.

$(X^4Z^3 - Z^3X^4)^T = (X^4Z^3)^T - (Z^3X^4)^T$

$= Z^3X^4 - X^4Z^3 = -(X^4Z^3 - Z^3X^4)$

Hence, skew symmetric.

Also, $(X^{23} + Y^{23})^T = -X^{23} - Y^{23}$

$= -(X^{23} + Y^{23})$, skew symmetric.

[Rating : Doable]

10. (b, c) : 1st solution

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ (2+\alpha)^2 & (2+2\alpha)^2 & (2+3\alpha)^2 \\ (3+\alpha)^2 & (3+2\alpha)^2 & (3+3\alpha)^2 \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - R_2$, $R_2 \to R_2 - R_1$, we get

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 5+2\alpha & 5+4\alpha & 5+6\alpha \end{vmatrix} = -648\alpha$$

Applying $R_3 \to R_3 - R_2$, we get

$$\begin{vmatrix} (1+\alpha)^2 & (1+2\alpha)^2 & (1+3\alpha)^2 \\ 3+2\alpha & 3+4\alpha & 3+6\alpha \\ 2 & 2 & 2 \end{vmatrix} = -648\alpha$$

Applying $C_3 \to C_3 - C_2$, $C_2 \to C_2 - C_1$, we get

Contd. on page no. 80

The Race to IITs

Top Rankers

1

SATVAT JAGWANI
Topper, IIT Kanpur Zone

The all-India first rank holder, son of a doctor couple, studied in Kota (Bansal Classes) to prepare for the JEE Advanced. An avid reader, he wants to take up computer science at IIT Bombay.

2

JANAK AGRAWAL
Topper, IIT Delhi Zone

Son of a business father and a homemaker mother, the 18-year-old studied for 15-16 hours a day to give the JEE Advanced his best shot. But no matter how busy he was, going to the temple every day was a must, says his sister Amrita, who is in the final year of her chartered accountancy. Agrawal's goal is to go to MIT and become a scientist. In his spare time, he likes to de-stress by playing video games, watching Hollywood movies and playing table tennis.

3

MUKESH PAREEKH

Mukesh Pareekh, 17-and-a-half, was born in a remote village in Rajasthan and plans to opt for computer science at IIT Bombay. Like others, he spent over 13 hours studying every day. His parents run a transport business and his brother Gopal says they were expecting a rank in the 20s but were pleasantly surprised when he made it to the top 10. This was his first attempt.

4

KAMANA NAGENDRA REDDY
Topper, IIT Madras Zone

Kamana Nagendra Reddy, 17, wants to study computer science and engineering at IIT Bombay. His routine included 12-13 hours of studying every day at the Chaitanya Narayana coaching college where his father is a professor. His mother is a homemaker.

5

BHARAT KHANDELWAL
Topper, IIT Bombay Zone

Khandelwal was a state-level footballer till Class X when he decided to concentrate on the IIT exam. His father is a CA, his mother, a homemaker and Khandelwal, like most other toppers, would like to get into IIT Bombay for computer science and engineering.

NUMBER OF SEATS
10,006
(Over 200 more than last year)

Number of IITs
18
Scores are also eligible for admission to Indian School of Mines, Dhanbad

Students who qualified

23,407 Male

3,049 Female

IIT Kharagpur has the most number of seats at
1,341
Followed by IIT, BHU at 1,090

Two IITs become functional this year:
Tirupati & Palakkad

A total of
26,456
students have qualified from among the 1.24 lakh for entry into the top IITs. This is 695 less than last year.

IIT Bombay
had the most number of qualifying candidates at **6,838**, followed by IIT Madras at **4,928**.

47

#1 among girls

KRATI TIWARI
Indore, Madhya Pradesh

10 **#1 in OBC-NCL**

MAJJI SANDEEP KUMAR

Majji Sandeep Kumar, 17, from Vizianagaram, cracked the exam in his first attempt. He studied at the Sri Chaitanya Narayana IIT Academy in Hyderabad for 13-14 hours a day. He was expecting a single-digit rank and says he missed it by one spot. His parents are government employees while his mother is a teacher, his father is local superintendent. He wants to study in the department of computer science at IIT-B.

Courtesy : ET

concept BOOSTERS

Class XI

Quadratic Equations

*ALOK KUMAR, B.Tech, IIT Kanpur

This column is aimed at Class XI students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

INTRODUCTION

- The standard form of the quadratic equation is $ax^2 + bx + c = 0$ where a, b, c are real numbers and $a \neq 0$.

ROOTS OF QUADRATIC EQUATION

- Let α, β be the roots of a quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0, a, b, c \in R)$

$$\alpha, \beta = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

(a) Sum of roots $= \alpha + \beta = \frac{-b}{a}$

(b) Product of roots $= \alpha \cdot \beta = \frac{c}{a}$

(c) Factorized form of $ax^2 + bx + c$ is $a(x - \alpha)(x - \beta)$

(d) If S be the sum and P be the product of roots, then quadratic equation is $x^2 - Sx + P = 0$.

NATURE OF ROOTS OF A QUADRATIC EQUATION

- Nature of roots of a quadratic equation $ax^2 + bx + c = 0$ (roots are real or complex)

The expression $b^2 - 4ac$ (called as discriminant, D)

(a) If $D > 0$ $(b^2 - 4ac > 0)$, then the roots are real and unequal.

(b) If $D = 0$ $(b^2 - 4ac = 0)$ then the roots are real and equal.

(c) If $D < 0$ $(b^2 - 4ac < 0)$, then the roots are non real *i.e* complex roots.

(d) If D *i.e.* $(b^2 - 4ac)$ is a perfect square and a, b and c are rational, then the roots are rational.

(e) If D *i.e* $(b^2 - 4ac)$ is not a perfect square but positive and a, b and c are rational, then roots are of the form $m + \sqrt{n}$ and $m - \sqrt{n}$.

(f) If $D < 0$ *i.e* $(b^2 - 4ac < 0)$, and the coefficient a, b and c are real then the roots are complex conjugate of each other *i.e.* roots are $p + iq$ and $p - iq$ $(p, q \in R \text{ and } i = \sqrt{-1})$

(g) If $a = 1$, $b, c \in I$ and the roots are rational numbers, then the roots must be integer.

(h) If a quadratic equation in x has more than two roots, then it is an identity in x (true for all real values of x) and $a = b = c = 0$

(i) The condition that a quadratic function $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$ may be resolved into two linear factors is that, $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ or

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

CONDITION FOR COMMON ROOT(S)

Consider two equations:

$ax^2 + bx + c = 0$ and $a'x^2 + b'x + c' = 0$

- **Two common roots:**

In such a case, two equations should be identical. The ratio of coefficients of x^2, x and x^0 must be same *i.e.*, $\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$.

- **One common root:**

Let α be the common root of two equations

$\Rightarrow a\alpha^2 + b\alpha + c = 0$ and $a'\alpha^2 + b'\alpha + c' = 0$

Solving the two equations

$$\frac{\alpha^2}{bc' - b'c} = \frac{\alpha}{a'c - ac'} = \frac{1}{ab' - a'b}$$

$$\Rightarrow \alpha = \frac{a'c - ac'}{ab' - a'b}, \quad \alpha^2 = \frac{bc' - b'c}{ab' - a'b}$$

$$\Rightarrow (bc' - b'c)(ab' - a'b) = (a'c - ac')^2$$

This is the condition for roots of two quadratic equations to be common.

*Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.

THE METHOD OF INTERVALS (WAVY CURVE METHOD)

- The method of intervals (or wavy curve) is used for solving inequalities of the form $\frac{P(x)}{Q(x)} \geq 0$, $\frac{P(x)}{Q(x)} \leq 0$ where $P(x)$ and $Q(x)$ are polynomials.

  If x_1 and x_2 $(x_1 < x_2)$ are two consecutive distinct roots of a polynomial equation, then within this interval the polynomial itself takes on values having the same sign.

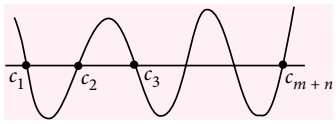
  Now find all the roots of the polynomial equations $P(x) = 0$ and $Q(x) = 0$

  Ignore the common roots and write

  $$\frac{P(x)}{Q(x)} = f(x) = \frac{(x-\alpha_1)(x-\alpha_2)....(x-\alpha_n)}{(x-\beta_1)(x-\beta_2)....(x-\beta_m)}.$$

  Where $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m$ are distinct real numbers. Then $f(x) = 0$ for $x = \alpha_1, \alpha_2, ..., \alpha_n$ and $f(x)$ is not defined for $x = \beta_1, \beta_2, .., \beta_m$. Apart from these $(m + n)$ real numbers $f(x)$ is either positive or negative. Now arrange $\alpha_1, \alpha_2, ..., \alpha_n, \beta_1, \beta_2, ..., \beta_m$ in an increasing order say $c_1, c_2, c_3, ..., c_{m+n}$. Plot them on real line. Draw a curve starting from right of c_{m+n} along the real line which alternately changes its sign at these points. This curve is known as the wavy curve.

  The intervals in which the curve is above the line will be the intervals for which $f(x)$ is positive and the intervals in which the curve is below the line will be the intervals in which $f(x)$ is negative.



QUADRATIC EXPRESSION

- The expression $ax^2 + bx + c$ is said to be a real quadratic expression in x where a, b, c are real and $a \neq 0$.

  Let $f(x) = ax^2 + bx + c$ where $a, b, c \in R$ $(a \neq 0)$

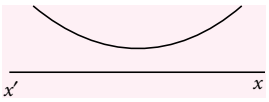
  $f(x)$ can be written as

  $$f(x) = a\left\{\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}\right\} = a\left\{\left(x+\frac{b}{2a}\right)^2 + \frac{-D}{4a^2}\right\}$$

  where $D = b^2 - 4ac$ is the discriminant of the quadratic expression. Then $y = f(x)$ represents a parabola whose axis is parallel to y-axis, with vertex at $A\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$.

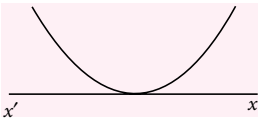
  Depending on the sign of a and $b^2 - 4ac$, $f(x)$ may be positive, negative or zero. This gives rise to the following cases:

1. $a > 0$ and $b^2 - 4ac < 0$ $\Leftrightarrow f(x) > 0 \ \forall \ x \in R$.

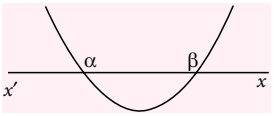
   In this case the parabola always remains above x-axis.
2. $a > 0$ and $b^2 - 4ac = 0$ $\Rightarrow f(x) \geq 0 \ \ \forall x \in R$.

   In this case the parabola touches the x-axis at one point and remains concave upwards.

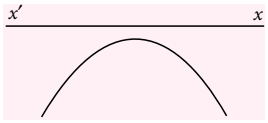
   

3. $a > 0$ and $b^2 - 4ac > 0$

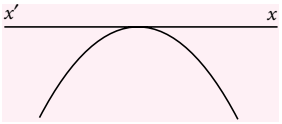
   Then $f(x) = 0$ have two real roots α, β $(\alpha < \beta)$. Then $f(x) > 0$ $\forall \ x \in (-\infty, \alpha) \cup (\beta, \infty)$

   And $f(x) < 0 \ \ \forall \ x \in (\alpha, \beta)$.
4. $a < 0$ and $b^2 - 4ac < 0$ $\Rightarrow f(x) < 0 \ \ \forall \ x \in R$

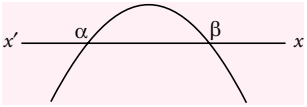
   

   In this case the parabola always remains below the x-axis.
5. $a < 0$ and $b^2 - 4ac = 0$ $\Rightarrow f(x) \leq 0 \ \ \forall \ x \in R$

   In this case parabola touches the x-axis and lies below the x-axis.
6. If $a < 0$ and $b^2 - 4ac > 0$

   Let $f(x) = 0$ have two real roots α and β $(\alpha < \beta)$.

   Then $f(x) < 0 \ \ \forall x \in (-\infty, \alpha) \cup (\beta, \infty)$ and $f(x) > 0 \ \ \forall \ x \in (\alpha, \beta)$

- **Interval in which the Roots lie**

  In few problems, the roots of the equation $ax^2 + bx + c = 0$ to lie in a given interval. For this we impose conditions on a, b and c.

1. If both the roots are positive *i.e.* they lie in $(0, \infty)$, then the sum of roots as well as the product of roots must be positive.

   $\Rightarrow \alpha + \beta = -\frac{b}{a} > 0$ and $\alpha\beta = \frac{c}{a} > 0$ with $b^2 - 4ac \geq 0$.

   If both the roots are negative *i.e* they lie in $(-\infty, 0)$ then the sum of the roots will be

negative and the product of roots must be positive.

$\Rightarrow \alpha+\beta=\frac{-b}{a}<0$ and $\alpha\beta=\frac{c}{a}>0$

with $b^2 - 4ac \geq 0$

2. Both the roots are greater than a given number k if the following three conditions are satisfied

$D \geq 0, -\frac{b}{2a} > k$ and $a \cdot f(k) > 0.$

3. Both the roots will be less than a given number k if the following conditions are satisfied

$D \geq 0, \frac{-b}{2a} < k$ and $a \cdot f(k) > 0$

4. Both the roots will lie in the given interval (k_1, k_2) if the following conditions are satisfied.

$D \geq 0,\ k_1 < \frac{-b}{2a} < k_2$ and $af(k_1) > 0,\ af(k_2) > 0$

5. Exactly one of the roots lie in the given interval (k_1, k_2) if $f(k_1) \cdot f(k_2) < 0$

6. A given number k will lie between the roots if $a \cdot f(k) < 0$. In particular, the roots of the equation will be of opposite signs if 0 lies between the roots $\Rightarrow a \cdot f(0) < 0$.

POLYNOMIALS

- Let $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are roots of the equation

$f(x) = a_0x^n + a_1x^{n-1} + a_2x^{n-2} + ... + a_{n-1}x + a_n = 0$

$\Rightarrow f(x) = a_0(x - \alpha_1)(x - \alpha_2) ... (x - \alpha_n)$

$\therefore\ a_0x^n + a_1x^{n-1} + ... + a_n = a_0(x - \alpha_1)(x - \alpha_2) ... (x - \alpha_n)$

Comparing the coefficients of x^{n-1} on both sides, we get

$S_1 = \alpha_1 + \alpha_2 + ... + \alpha_n = \sum \alpha_i = -\frac{a_1}{a_0}$

$S_1 = -\frac{\text{coeff. of } x^{n-1}}{\text{coeff. of } x^n}$

Comparing the coefficients of x^{n-2} on both sides, we get

$S_2 = \alpha_1\alpha_2 + \alpha_1\alpha_3 + ... = \sum_{i \neq j} \alpha_i \alpha_j = (-1)^2 \frac{a_2}{a_0}$

$S_2 = \frac{(-1)^2 \text{ coeff. of } x^{n-2}}{\text{coeff. of } x^n}$

.....

.....

$S_n = \alpha_1\alpha_2\alpha_3...\alpha_n = (-1)^n \frac{a_n}{a_0} = \frac{(-1)^n \text{ const.term}}{\text{coeff. of } x^n}$

Hence S_k denotes the sum of the products of the roots taken k at a time.

- **Particular Cases**

(a) Quadratic equation: If α, β are the roots of quadratic equation $ax^2 + bx + c = 0$ then

$\alpha+\beta=\frac{-b}{a}$ and $\alpha \cdot \beta=\frac{c}{a}$

(b) Cubic equation: If α, β and γ are the roots of cubic equation $ax^3 + bx^2 + cx + d = 0$ then

$\alpha+\beta+\gamma=\frac{-b}{a},\ \alpha\beta+\beta\gamma+\gamma\alpha=(-1)^2\frac{c}{a}=\frac{c}{a}$

$\alpha\beta\gamma=(-1)^3\frac{d}{a}=\frac{-d}{a}$

PROBLEMS

SECTION-I

Single Correct Answer Type

1. If α, β are the roots of the equation $ax^2 + bx + c = 0$ and $S_n = \alpha^n + \beta^n$ $(n \geq 2)$, then $aS_{n+1} + bS_n + cS_{n-1} =$

(a) 0 (b) $a + b + c$

(c) $(a + b + c)^n$ (d) $n^2 abc$

2. The number of solutions of $|[x] - 2x| = 4$, where $[x]$ is the greatest integer $\leq x$ is

(a) 2 (b) 4 (c) 1 (d) infinite

3. Let a, b and c be real numbers such that $a + 2b + c = 4$ then the maximum value of $ab + bc + ca$ is

(a) 1 (b) 2 (c) 3 (d) 4

4. If α, β, γ are the roots of the equation $x^3 + px + q = 0$, then the value of the determinant $\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix}$ is

(a) 4 (b) 2 (c) 0 (d) –2

5. The number of points (p, q) such that $p, q \in \{1, 2, 3, 4\}$ and the equation $px^2 + qx + 1 = 0$ has real roots is

(a) 7 (b) 8

(c) 9 (d) None of these

6. The values of b and c for which the identity $f(x + 1) - f(x) = 8x + 3$ is satisfied, where $f(x) = bx^2 + cx + d$ are

(a) $b = 2, c = 1$ (b) $b = 4, c = -1$

(c) $b = -1, c = 4$ (d) $b = -1, c = 1$

7. Let $f(x) = ax^2 + bx + c$, $g(x) = ax^2 + px + q$, where $a, b, c, q, p \in R$ and $b \neq p$. If their discriminants are equal and $f(x) = g(x)$ has a root α, then

(a) α will be A.M. of the roots of $f(x) = 0$, $g(x) = 0$

(b) α will be G.M of all the roots of $f(x) = 0$, $g(x) = 0$

(c) α will be A.M of the roots of $f(x) = 0$ or $g(x) = 0$

(d) α will be G.M of the roots of $f(x) = 0$ or $g(x) = 0$

8. If the equations $x^2 + 2\lambda x + \lambda^2 + 1 = 0$, $\lambda \in R$ and $ax^2 + bx + c = 0$ where a, b, c are lengths of sides of triangle have a common root, then the possible range of values of λ is

(a) $(0, 2)$ (b) $(\sqrt{3}, 3)$

(c) $(2\sqrt{2}, 3\sqrt{2})$ (d) $(0, \infty)$

9. The number of real or complex solutions of $x^2 - 6|x| + 8 = 0$ is

(a) 6 (b) 7 (c) 8 (d) 9

10. The number of solutions of $\sqrt{3x^2 + x + 5} = x - 3$ is

(a) 0 (b) 1 (c) 2 (d) 4

11. The value of a for which one root of the quadratic equation $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ is twice as large as other, is

(a) $-2/3$ (b) $1/3$ (c) $-1/3$ (d) $2/3$

12. If $f(x) = x^2 + 2bx + 2c^2$ and $g(x) = -x^2 - 2cx + b^2$ are such that min $f(x) >$ max $g(x)$, then relation between b and c is

(a) no relation (b) $0 < c < b/2$

(c) $|c| < \frac{|b|}{\sqrt{2}}$ (d) $|c| > \sqrt{2}\,|b|$

13. The equation $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ in variable x has real roots, if p belongs to the interval

(a) $(0, 2\pi)$ (b) $(-\pi, 0)$

(c) $(-\pi/2, \pi/2)$ (d) $(0, \pi)$

14. If $x^2 + 2ax + 10 - 3a > 0$ for each $x \in R$, then

(a) $a < -5$ (b) $-5 < a < 2$

(c) $a > 5$ (d) $2 < a < 5$

15. Sum of all the values of x satisfying the equation $\log_{17} \log_{11}(\sqrt{x+11} + \sqrt{x}) = 0$ is

(a) 25 (b) 36 (c) 171 (d) 0

16. The number of solutions of the equation $x^2 + [x] - 4x + 3 = 0$ is (where $[\cdot]$ denotes G.I.F.)

(a) 0 (b) 1 (c) 2 (d) 3

17. If a, b, c are positive numbers such that $a > b > c$ and the equation $(a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b) = 0$ has a root in the interval $(-1, 0)$, then

(a) b cannot be the G.M. of a, c

(b) b may be the G.M. of a, c

(c) b is the G.M. of a, c

(d) none of these

18. Let $f(x)$ be a function such that $f(x) = x - [x]$, where $[x]$ is the greatest integer less than or equal to x. Then the number of solutions of the equation $f(x) + f\left(\frac{1}{x}\right) = 1$ is (are)

(a) 0 (b) 1 (c) 2 (d) infinite

19. If both the roots of $(2a - 4)9^x - (2a - 3)3^x + 1 = 0$ are non-negative, then

(a) $0 < a < 2$ (b) $2 < a < \frac{5}{2}$

(c) $a < \frac{5}{4}$ (d) $a > 3$

20. If the equation $x^2 + 9y^2 - 4x + 3 = 0$ is satisfied for real values of x and y then

(a) $x \in [1, 3], y \in [1, 3]$

(b) $x \in [1, 3], \; y \in \left[\frac{-1}{3}, \frac{1}{3}\right]$

(c) $x \in \left[\frac{-1}{3}, \frac{1}{3}\right], \; y \in [1, 3]$

(d) $x \in \left[\frac{-1}{3}, \frac{1}{3}\right], \; y \in \left[\frac{-1}{3}, \frac{1}{3}\right]$

21. The equation $a_8x^8 + a_7x^7 + a_6x^6 + \ldots + a_0 = 0$ has all its roots positive and real (where $a_8 = 1$, $a_7 = -4$, $a_0 = 1/2^8$), then

(a) $a_1 = \frac{1}{2^8}$ (b) $a_1 = -\frac{1}{2^4}$

(c) $a_2 = \frac{7}{2^5}$ (d) $a_2 = \frac{7}{2^8}$

22. If $f(x)$ is a polynomial of least degree such that $f(r) = \frac{1}{r}$, $r = 1, 2, 3, \ldots\ldots, 9$, then $f(10) =$

(a) 1 (b) 1/2 (c) 1/10 (d) 1/5

23. If a, b, c be the sides of a triangle ABC and if roots of the equation $a(b - c)x^2 + b(c - a)x + c(a - b) = 0$ are equal, then $\sin^2\left(\frac{A}{2}\right), \sin^2\left(\frac{B}{2}\right), \sin^2\left(\frac{C}{2}\right)$ are in

(a) A.P. (b) G.P. (c) H.P. (d) A.G.P.

SECTION-II

Multiple Correct Answer Type

24. Consider the fraction $\frac{x^3 - ax^2 + 19x - a - 4}{x^3 - (a+1)x^2 + 23x - a - 7}$

(a) The value of 'a' at which the above fraction admits of reduction is 8

(b) The value of 'a' at which the above fraction admits of reduction is 4

(c) The lowest admitted reduction form of the fraction is $\frac{x-4}{x-5}$

(d) The lowest admitted reduction form of the fraction is $\frac{x-3}{x-4}$

25. Two numbers are such that their sum multiplied by the sum of their squares is 5500 and their difference multiplied by the difference of the squares is 352. Then the numbers are
(a) Prime numbers only
(b) Odd positive integers
(c) Prime but not odd
(d) Odd but not prime

26. If by eliminating x between the equations $x^2 + ax + b = 0$ and $xy + l(x + y) + m = 0$, a quadratic equation in y is formed whose roots are the same as those original quadratic in x, then
(a) $a = 2l$ (b) $b = m$
(c) $b + m = al$ (d) $a + b = l$

27. Let $|a| < |b|$ and a, b are the roots of the equation $x^2 - |\alpha|x - |\beta| = 0$. If $|\alpha| < b - 1$, then the equation $\log_{|a|}\left(\frac{x}{b}\right)^2 - 1 = 0$ has at least one
(a) root lying between $(-\infty, a)$
(b) root lying between (b, ∞)
(c) negative root
(d) positive root

28. The value of x satisfying the equation
$x^4 - 2\left(x\sin\left(\frac{\pi}{2}x\right)\right)^2 + 1 = 0$ is
(a) 1 (b) -1
(c) 0 (d) No value of x

29. Let $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$, then $f(x) = 0$ has
(a) exactly one real root in (2, 3)
(b) exactly one real root in (3, 4)
(c) at least one real root in (2, 3)
(d) None of these

30. If $x_1 > x_2 > x_3$ and x_1, x_2, x_3 are roots of $\frac{x-a}{b} + \frac{x-b}{a} = \frac{b}{x-a} + \frac{a}{x-b}$; $(a, b > 0)$ and $x_1 - x_2 - x_3 = c$, then a, c, b are in
(a) A.P. (b) G.P.
(c) H.P. (d) None of these

31. If the equation whose roots are the squares of the roots of the cubic $x^3 - ax^2 + bx - 1 = 0$ is identical with the given cubic equation, then
(a) a, b are roots of $x^2 + x + 2 = 0$
(b) $a = b = 0$ (c) $a = b = 3$
(d) $a = 0$, $b = 3$

32. $\frac{\pi^e}{x-e} + \frac{e^\pi}{x-\pi} + \frac{\pi^\pi + e^e}{x-\pi-e} = 0$ has
(a) One real root in (e, π) and other in $(\pi - e, e)$
(b) One real root in (e, π) and other in $(\pi, \pi + e)$
(c) Two real roots in $(\pi - e, \pi + e)$
(d) No real root

SECTION-III

Comprehension Type

Paragraph for Question No. 33 to 35

Let x_1, x_2, x_3, x_4 be the roots (real or complex) of the equation $x^4 + ax^3 + bx^2 + cx + d = 0$.
If $x_1 + x_2 = x_3 + x_4$ and $a, b, c, d \in R$, then

33. If $a = 2$, then the value of $b - c$ is
(a) -1 (b) 1 (c) -2 (d) 2

34. If $b < 0$ then how many different real values of 'a' we may have?
(a) 3 (b) 2 (c) 1 (d) 0

35. If $b + c = 1$ and $a \neq -2$, then for real values of 'a', the value of $c \in$
(a) $\left(-\infty, \frac{1}{4}\right)$ (b) $(-\infty, 3)$
(c) $(-\infty, 1)$ (d) $(-\infty, 4)$

Paragraph for Question No. 36 to 38

If roots of the equation $x^4 - 12x^3 + bx^2 + cx + 81 = 0$ are positive, then

36. Value of b is
(a) -54 (b) 54 (c) 27 (d) -27

37. Value of c is
(a) 108 (b) -108 (c) 54 (d) -54

38. Root of equation $2bx + c = 0$ is
(a) $-1/2$ (b) $1/2$ (c) 1 (d) -1

Paragraph for Question No. 39 to 41

$P(x)$ be polynomial of degree at most 5 which leaves remainders -1 and 1 upon division by $(x - 1)^3$ and $(x + 1)^3$ respectively.

39. Numbers of real roots of $P(x) = 0$
(a) 1 (b) 3 (c) 5 (d) 2

40. The maximum value of $y = P''(x)$ can be obtained at $x =$
(a) $-\frac{1}{\sqrt{3}}$ (b) 0 (c) $\frac{1}{\sqrt{3}}$ (d) 1

41. The sum of pairwise product of all roots (real and complex) of $P(x) = 0$ is
(a) $-5/3$ (b) $-10/3$ (c) 2 (d) -5

Paragraph for Question No. 42 to 44

Let $f(x) = x^4 + ax^3 + bx^2 + ax + 1$ be a polynomial where a and b are real numbers, then

42. If $f(x) = 0$ has two different pairs of equal roots, then the value of $a + b$ is
(a) 0 (b) -4 (c) -2 (d) 4

43. If $f(x) = 0$ has two different negative roots and two equal positive roots, then the least integral value of a is
(a) 1 (b) 2 (c) 3 (d) 4

44. If all the roots are imaginary and $b = -1$ then number of all possible integral values of a is
(a) 0 (b) 1 (c) 2 (d) 4

Paragraph for Question No. 45 to 47

$y = ax^2 + bx + c = 0$, $\forall$ $a, b, c \in R$ with $a \neq 0$ is a quadratic equation which has real roots if and only if $b^2 - 4ac \geq 0$. If $F(x, y) = 0$ is a second degree equation, then using above fact we can get the range of x and y by treating it as quadratic equation in y or x. Similarly $ax^2 + bx + c \geq 0$ $\forall$ $x \in R$ if $a > 0$ and $b^2 - 4ac \leq 0$.

45. If $0 < \alpha, \beta < 2\pi$, then the number of ordered pairs (α, β) satisfying $\sin^2(\alpha + \beta) - 2\sin\alpha \sin(\alpha + \beta) + \sin^2\alpha + \cos^2\beta = 0$ is
(a) 2 (b) 0 (c) 4 (d) 6

46. Let x, y, z be real variables satisfying the equations $x + y + z = 6$ and $xy + yz + zx = 7$, then the range of x is

(a) $\left[\frac{6-\sqrt{5}}{3}, \frac{6+\sqrt{15}}{3}\right]$ (b) $\left[\frac{6-2\sqrt{15}}{3}, \frac{6+2\sqrt{15}}{3}\right]$

(c) $\left[\frac{6-\sqrt{15}}{2}, \frac{6+\sqrt{15}}{2}\right]$ (d) $\left[\frac{6-\sqrt{15}}{7}, \frac{6+2\sqrt{15}}{7}\right]$

47. If $9^{x+1} + (a^2 - 4a - 2)3^x + 1 > 0$ $\forall$ $x \in R$, then
(a) $a \in R$ (b) $a \in R^+$
(c) $a \in [1, \infty)$ (d) $a \in R - \{2\}$

SECTION-IV

Matrix-Match Type

48. Match the statements of Column I with values of Column II.

Column-I		Column-II	
(A)	The least positive integral values of λ for which $(\lambda - 2)x^2 + 8x + (\lambda + 4) > 0$, for all real x is	(p)	3
(B)	The equation $x^2 + 2(a^2 + 1)x + (a^2 - 14a + 48) = 0$ possesses roots of opposite signs then value of 'a' can be	(q)	5
(C)	If the equation $ax^2 + 2bx + 4c = 16$ has no real roots and $a + c > b + 4$ then integral value of c can be equal to	(r)	7
(D)	If N be the number of solution of the equation $\lvert x^2 - x - 6\rvert = x + 2$ then the value of N is	(s)	12
		(t)	20

49. Match the statements/expressions in Column I with the open intervals in Column II.

Column-I		Column-II	
(A)	If $a, b > 0$ and $a \cdot b = 2a + 3b$, then minimum value of ab is	(p)	-1
(B)	Number of real roots of equation $x^2 - 4x + 6 = 2\sin\left(\frac{\pi x}{4}\right)$ is/are	(q)	0
(C)	The equation $x^3 - 6x^2 + 9x + \lambda = 0$ have exactly one root in (1, 3) then $[\lambda + 3]$ is, where $[\cdot]$ is GIF	(r)	1
(D)	If $x^2 + 3\lambda x + 2 = 0$ & $(b - c)x^2 + (c - a)x + (a - b) = 0$ have both roots common then $[\lambda + 1]$ is	(s)	2
		(t)	24

50. Match the following

Column-I		Column-II	
(A)	If $a^2 - 4a - 3 = 0$, then the value of $\frac{a^3 - a^2 + a - 1}{a^2 - 1}(a^2 \neq 1) =$	(p)	3
(B)	The number of value(s) of x satisfying the equation $\sqrt[4]{\lvert x-3\rvert^{x+1}} = \sqrt[3]{\lvert x-3\rvert^{x-2}}$ is	(q)	2
(C)	The number of value(s) of x satisfying the equation $3^x + 1 - \lvert 3^x - 1\rvert = 2\log_5\lvert 6 - x\rvert$ is	(r)	4
(D)	If the sum of the first $2n$ terms of the A.P., 2, 5, 8,……, is equal to the sum of the first n terms of the A.P., 57, 59, 61,…, then n equals	(s)	11

SECTION-V

Integer Answer Type

51. If λ is the minimum value of the expression $|x - p| + |x - 15| + |x - p - 15|$ for x in the range $p \leq x \leq 15$ where $0 < p < 15$. Then $\frac{\lambda}{5} =$

52. Let $P(x) = x^2 + bx + c$, where b and c are integers. If $P(x)$ is a factor of both $x^4 + 6x^2 + 25$ and $3x^4 + 4x^2 + 28x + 5$, find the value of $P(1)$.

53. Let a, b, c be the three roots of the equation $x^3 + x^2 - 333x - 1002 = 0$. If $P = a^3 + b^3 + c^3$ then the value of $\frac{P}{2006} =$

54. The number of the distinct real roots of the equation $(x+1)^5 = 2(x^5+1)$ is

55. The equation $2(\log_3 x)^2 - |\log_3 x| + a = 0$ has exactly four real solutions if $a \in \left(0, \frac{1}{K}\right)$, then the value of K is

56. Let α, β and γ be the roots of equation $f(x) = 0$, where $f(x) = x^3 + x^2 - 5x - 1$. Then the value of $|[\alpha] + [\beta] + [\gamma]|$, where $[\cdot]$ denotes the greatest integer function, is equal to

57. The set of real parameter 'a' for which the equation $x^4 - 2ax^2 + x + a^2 - a = 0$ has all real solutions, is given by $\left[\frac{m}{n}, \infty\right)$ where m and n are relatively prime positive integers, then the value of $(m+n)$ is

58. Number of positive integer n for which $n^2 + 96$ is a perfect square is

59. If α, β be the roots of $x^2 + px - q = 0$ and γ, δ are the roots of $x^2 + px + r = 0$, $q + r \neq 0$, then $\frac{(\alpha-\gamma)(\alpha-\delta)}{(\beta-\gamma)(\beta-\delta)}$ is equal to

60. Let $p(x) = x^5 + x^2 + 1$ have roots x_1, x_2, x_3, x_4 and x_5, $g(x) = x^2 - 2$, then the value of $g(x_1)g(x_2)g(x_3)g(x_4)g(x_5) - 30g(x_1 x_2 x_3 x_4 x_5)$, is

SOLUTIONS

1. (a) : $S_{n+1} = \alpha^{n+1} + \beta^{n+1}$
$= (\alpha+\beta)(\alpha^n + \beta^n) - \alpha\beta(\alpha^{n-1} + \beta^{n-1})$
$= -\frac{b}{a}\cdot S_n - \frac{c}{a}\cdot S_{n-1}$

2. (b) : If $x = n \in Z$, $|n - 2n| = 4 \Rightarrow n = \pm 4$
If $x = n + k$ where $0 < k < 1$ then $|n - 2(n+k)| = 4$, is possible if $k = 1/2 \Rightarrow |-n - 1| = 4$
$\therefore \quad n = 3, -5$

3. (d) : Let $ab + bc + ca = x$
$\Rightarrow 2b^2 + 2(c-2)b - 4c + c^2 + x = 0$
Since $b \in R$, $\therefore \quad c^2 - 4c + 2x - 4 \le 0$
Since $c \in R$, $\therefore \quad x \le 4$

4. (c) : Since α, β, γ are the roots of $x^3 + px + q = 0$
$\therefore \quad \alpha + \beta + \gamma = 0$
Applying $C_1 \to C_1 + C_2 + C_3$, then

$$\begin{vmatrix} \alpha+\beta+\gamma & \beta & \gamma \\ \alpha+\beta+\gamma & \gamma & \alpha \\ \alpha+\beta+\gamma & \alpha & \beta \end{vmatrix} = \begin{vmatrix} 0 & \beta & \gamma \\ 0 & \gamma & \alpha \\ 0 & \alpha & \beta \end{vmatrix} = 0$$

5. (a) : $px^2 + qx + 1 = 0$ has real roots if $q^2 - 4p \ge 0$ or $q^2 \ge 4p$
Since $p, q \in \{1, 2, 3, 4\}$
The required points are (1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (4, 4).
So, the required number is 7

6. (b) : $\because f(x+1) - f(x) = 8x + 3$
$\Rightarrow \{b(x+1)^2 + c(x+1) + d\} - \{bx^2 + cx + d\} = 8x + 3$
$\Rightarrow b\{(x+1)^2 - x^2\} + c = 8x + 3$
$\Rightarrow b(2x+1) + c = 8x + 3$
On comparing, we get
$2b = 8$ and $b + c = 3$
Then, $b = 4$ and $c = -1$

7. (a) : $a\alpha^2 + b\alpha + c = a\alpha^2 + p\alpha + q$

$$\Rightarrow \alpha = \frac{q-c}{b-p} \qquad \ldots (i)$$

And $b^2 - 4ac = p^2 - 4aq$
$\Rightarrow b^2 - p^2 = 4a(c-q)$

$$\Rightarrow b + p = \frac{4a(c-q)}{b-p} = -4a\alpha \qquad \text{(from (i))}$$

$$\Rightarrow \alpha = \frac{-(b+p)}{4a} = \frac{\frac{-b}{a} - \frac{p}{a}}{4}$$

which is A.M. of all the roots of $f(x) = 0$ and $g(x) = 0$.

8. (a) : $(x+\lambda)^2 + 1 = 0$ has clearly imaginary roots
So, both roots of the equations are common

$$\therefore \quad \frac{a}{1} = \frac{b}{2\lambda} = \frac{c}{\lambda^2+1} = k \text{ (say)}$$

Then $a = k$, $b = 2\lambda k$, $c = (\lambda^2 + 1)k$
As a, b, c are sides of a triangle
$\therefore \quad a + b > c \Rightarrow 2\lambda + 1 > \lambda^2 + 1 \Rightarrow \lambda^2 - 2\lambda < 0$
$\Rightarrow \lambda \in (0, 2)$

9. (a) : If x is real,
$x^2 - 6|x| + 8 = 0 \Rightarrow |x|^2 - 6|x| + 8 = 0$
$\Rightarrow |x| = 2, 4 \Rightarrow x = \pm 2, \pm 4$
If x is non-real, say $x = \alpha + i\beta$, then

$$(\alpha+i\beta)^2 - 6\sqrt{\alpha^2+\beta^2} + 8 = 0 \quad (|\alpha+i\beta| = \sqrt{\alpha^2+\beta^2})$$

$$\Rightarrow (\alpha^2 - \beta^2 + 8 - 6\sqrt{\alpha^2+\beta^2}) + 2i\alpha\beta = 0$$

Comparing real and imaginary parts,
$\alpha\beta = 0 \Rightarrow \alpha = 0$ (if $\beta = 0$ then x is real)
and $-\beta^2 + 8 - 6\sqrt{\beta^2} = 0$

$$\beta^2 \pm 6\beta - 8 = 0 \Rightarrow \beta = \frac{\mp 6 \pm \sqrt{68}}{2}$$

i.e., $\beta = \pm(3 - \sqrt{17})$
Hence $\pm(3-\sqrt{17})i$ are non-real roots.

10. (a) : We must have
$3x^2 + x + 5 \ge 0$ and $x - 3 \ge 0$ or $x \ge 3$
We have, $\sqrt{3x^2 + x + 5} = x - 3 \qquad \ldots (i)$

Squaring (i) both sides, we get

$3x^2 + x + 5 = x^2 - 6x + 9$

$\Rightarrow 2x^2 + 7x - 4 = 0 \Rightarrow (2x - 1)(x + 4) = 0$

$\Rightarrow x = 1/2, -4$

None of these satisfy the inequality $x \geq 3$. Thus, (i) has no solution.

11. (d) : $(a^2 - 5a + 3)x^2 + (3a - 1)x + 2 = 0$ ...(i)

Let α and 2α be the roots of (1), then

$(a^2 - 5a + 3)\alpha^2 + (3a - 1)\alpha + 2 = 0$ ...(ii)

and $(a^2 - 5a + 3)(4\alpha^2) + (3a - 1)(2\alpha) + 2 = 0$ ...(iii)

Multiplying (ii) by 4 and subtracting it from (iii), we get $(3a - 1)(2\alpha) + 6 = 0$

Clearly $a \neq 1/3$. Therefore, $\alpha = -3/(3a - 1)$

Putting this value in (ii), we get

$(a^2 - 5a + 3)(9) - (3a - 1)^2(3) + 2(3a - 1)^2 = 0$

$\Rightarrow 9a^2 - 45a + 27 - (9a^2 - 6a + 1) = 0$

$\Rightarrow -39a + 26 = 0 \Rightarrow a = 2/3$

For $a = 2/3$, the equation becomes $x^2 + 9x + 18 = 0$ whose roots are $-3, -6$.

12. (d) : $f(x) = (x + b)^2 + 2c^2 - b^2$

$\Rightarrow \min f(x) = 2c^2 - b^2$

Also $g(x) = -x^2 - 2cx + b^2 = b^2 + c^2 - (x + c)^2$

$\Rightarrow \max g(x) = b^2 + c^2$

As $\min f(x) > \max g(x)$, we get $2c^2 - b^2 > b^2 + c^2$

$\Rightarrow c^2 > 2b^2 \Rightarrow |c| > \sqrt{2}|b|$

13. (d) : $(\cos p - 1)x^2 + (\cos p)x + \sin p = 0$ ...(i)

Discriminant of (i) is given by

$D = \cos^2 p - 4(\cos p - 1)\sin p = \cos^2 p + 4(1 - \cos p)\sin p$

Note that $\cos^2 p \geq 0$, $1 - \cos p \geq 0$.

Thus, $D \geq 0$ if $\sin p \geq 0$ *i.e.* if $p \in (0, \pi)$.

14. (b) : $x^2 + 2ax + 10 - 3a > 0 \ \forall \ x \in R$

$\Rightarrow D = 4a^2 - 4(10 - 3a) < 0$

$\Rightarrow a^2 + 3a - 10 < 0 \Rightarrow (a + 5)(a - 2) < 0$

$\Rightarrow -5 < a < 2$

15. (a) : $\log_{17} \log_{11}(\sqrt{x+11} + \sqrt{x}) = 0$ ...(i)

Equation (1) is defined if $x \geq 0$

We can rewrite (i) as $\log_{11}(\sqrt{x+11} + \sqrt{x}) = 17^0 = 1$

$\Rightarrow \sqrt{x+11} + \sqrt{x} = 11^1 = 11$

$\Rightarrow \sqrt{x+11} = 11 - \sqrt{x}$ ...(ii)

Squaring (ii) both sides, we get

$x + 11 = 121 - 22\sqrt{x} + x$

$\Rightarrow 22\sqrt{x} = 110 \Rightarrow \sqrt{x} = 5$ or $x = 25$

This clearly satisfies (i). Thus, sum of all the values satisfying (i) is 25.

16. (a) : Given equation can be written as

$(x^2 - 3x + 3) - f = 0$ where $f = x - [x]$ and $0 \leq f < 1$

$\therefore \ 0 \leq x^2 - 3x + 3 < 1$

Solving $x^2 - 3x + 3 = 0$; roots are imaginary

$\therefore \ x^2 - 3x + 3 \geq 0 \ \forall \ x \in R$

Solving $x^2 - 3x + 3 < 1 \Rightarrow 1 < x < 2$

if $1 < x < 2$; $[x] = 1$

Putting $[x] = 1$ in the given equation and solving we get $x = 2$. But $1 < x < 2$

$\therefore$ the given equation has no solution.

17. (a) : Let $f(x) = (a + b - 2c)x^2 + (b + c - 2a)x + (c + a - 2b)$

According to the given condition, we have

$f(0) f(-1) < 0$

i.e. $(c + a - 2b)(2a - b - c) < 0$

i.e. $(c + a - 2b)(a - b + a - c) < 0$

i.e. $c + a - 2b < 0$

$[a > b > c$, given $\Rightarrow a - b > 0, a - c > 0]$

i.e. $b > \dfrac{a+c}{2}$

$\Rightarrow$ b cannot be the G.M. of a and c, since G.M. < A.M. always.

18. (d)

19. (b) : Putting $3^x = y$, we have

$(2a - 4)y^2 - (2a - 3)y + 1 = 0$

This equation must have real solution

$\Rightarrow (2a - 3)^2 - 4(2a - 4) \geq 0$

$\Rightarrow 4a^2 - 20a + 25 \geq 0$

$\Rightarrow (2a - 5)^2 \geq 0$, which is true.

$y = 1$ satisfies the equation

Since 3^x is positive and $3^x \geq 3^0 \Rightarrow y \geq 1$

Product of the roots $= (1 \times y) > 1$

$\Rightarrow \dfrac{1}{2a-4} > 1 \Rightarrow 2a - 4 < 1 \Rightarrow a < \dfrac{5}{2}$

Sum of the roots $= \dfrac{2a-3}{2a-4} > 1$

$\Rightarrow \dfrac{(2a-3)-(2a-4)}{2a-4} > 0$

$\Rightarrow \dfrac{1}{2a-4} > 0 \Rightarrow a > 2 \Rightarrow 2 < a < \dfrac{5}{2}$

20. (b) : Given equation is

$x^2 + 9y^2 - 4x + 3 = 0$ ...(i)

or, $x^2 - 4x + 9y^2 + 3 = 0$

Since x is real $\therefore \ (-4)^2 - 4(9y^2 + 3) \geq 0$

or $16 - 4(9y^2 + 3) \geq 0$ or $4 - 9y^2 - 3 \geq 0$

or $9y^2 - 1 \leq 0$ or $9y^2 \leq 1$ or $y^2 \leq \dfrac{1}{9}$

Now, $y^2 \leq \dfrac{1}{9} \Leftrightarrow -\dfrac{1}{3} \leq y \leq \dfrac{1}{3}$ ...(ii)

Equation (i) can also be written as

$9y^2 + 0y + x^2 - 4x + 3 = 0$ ...(iii)

Since y is real $\therefore \ 0^2 - 4 \cdot 9(x^2 - 4x + 3) \geq 0$

or $x^2 - 4x + 3 \leq 0 \Rightarrow x \in [1, 3]$

21. (b) : Let the roots be $\alpha_1, \alpha_2, ..., \alpha_8$

$\Rightarrow \alpha_1 + \alpha_2 + ... + \alpha_8 = 4$

$\alpha_1\alpha_2 .... \alpha_8 = \frac{1}{2^8}$

$\Rightarrow (\alpha_1\alpha_2 .... \alpha_8)^{1/8} = \frac{1}{2} = \frac{\alpha_1 + \alpha_2 + ... + \alpha_8}{8}$

$\Rightarrow$ A.M. = G.M. $\Rightarrow$ all the roots are equal to 1/2.

$\Rightarrow a_1 = -{}^8C_7\left(\frac{1}{2}\right)^7 = -\frac{1}{2^4}$

$a_2 = {}^8C_6\left(\frac{1}{2}\right)^6 = -\frac{7}{2^4}$

22. (d) : $xf(x) - 1 = 0$ has roots 1, 2, 3, ..., 9

$xf(x) - 1 = A(x - 1)(x - 2) .... (x - 9)$

Put $x = 0 \Rightarrow A = \frac{1}{9!}$

Put $x = 10 \Rightarrow 10\, f(10) - 1 = 1 \Rightarrow f(10) = \frac{1}{5}$

23. (c)

24. (a, c) : Subtracting numerator from denominator, we get

$x^2 - 4x + 3$ *i.e.* $(x - 1)(x - 3)$

Thus it is concluded that numerator and denominator must be completely divisible by $(x - 1)$ or $(x - 3)$. In other words, both must vanish when $x = 1$ or when $x = 3$, if $x = 3$ or 1 we get, $a = 8$

And fraction becomes

$$\frac{x^3 - 8x^2 + 19x - 12}{x^3 - 9x^2 + 23x - 15} = \frac{x^2 - 7x + 12}{x^2 - 8x + 15} = \frac{x - 4}{x - 5}$$

25. (b) : Let the two numbers be x, y then

$(x + y)(x^2 + y^2) = 5500$ ... (i)

and $(x - y)(x^2 - y^2) = 352$ ... (ii)

After solving these two equations

$x = 9$ or 13 ; $y = 13$ or 9

26. (a, b, c)

27. (a, b, c, d) : $|\alpha|$ = sum of roots = $b + a$

$-|\beta|$ = product of roots = ab

Because $|a| < |b|$ so a is negative and b is positive.

Now, $|\alpha| < b - 1 \Rightarrow a + b < b - 1 = a < -1$

Because a is negative so magnitude of 'a' is greater than one and magnitude of b is greater than $1 + |\alpha|$ or say greater than 2.

Now, $\log_{|a|}\left(\frac{x}{b}\right)^2 - 1 = 0 \Rightarrow \left(\frac{x}{b}\right)^2 = |a|$

$\Rightarrow x = \pm b\sqrt{|a|}$

$\Rightarrow$ one root lies in $(-\infty, a)$ and other root lies in (b, ∞)

28. (a, b) : $x^4 - 2\left(x\sin\left(\frac{\pi}{2}x\right)\right)^2 + 1 = 0$

$\Rightarrow x^4 + 1 = 2x^2\sin^2\left(\frac{\pi}{2}x\right)$

$\Rightarrow x^2 + \frac{1}{x^2} = 2\sin^2\left(\frac{\pi}{2}x\right)$

Now, L.H.S ≥ 2 where as R.H.S. ≤ 2

So, equality holds when

$x^2 + \frac{1}{x^2} = 2$ and $2\sin^2\left(\frac{\pi}{2}x\right) = 2 \Rightarrow x = \pm 1$

29. (a, b) : $f(x) = \frac{3}{x-2} + \frac{4}{x-3} + \frac{5}{x-4}$

$\because$ $f(2^+) \to \infty$ and $f(3^-) \to -\infty$

$\Rightarrow f(x) = 0$ has exactly one root in (2, 3)

Again, $\because$ $f(3^+) \to \infty$ and $f(4^-) \to -\infty$

$\Rightarrow f(x) = 0$ has exactly one root in (3, 4).

30. (c)

31. (a, b, c) : If roots of the equation be α, β, γ then

$\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= a^2 - 2b$

$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2$

$= (\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta\gamma) \times (\alpha + \beta + \gamma)$

$= b^2 - 2a$

$\alpha^2\beta^2\gamma^2 = 1$

So, the equation whose roots are $\alpha^2, \beta^2, \gamma^2$ is

$x^3 - (a^2 - 2b)x^2 + (b^2 - 2a)x - 1 = 0$

It is identical to $x^3 - ax^2 + bx + 1 = 0$

$\therefore$ $a^2 - 2b = a$ and $b^2 - 2a = b$, eliminating b, we get

$\frac{(a^2 - a)^2}{4} - 2a = \frac{a^2 - a}{2}$

$\Rightarrow a\{a(a - 1)^2 - 8 - 2(a - 1)\} = 0$

$\Rightarrow a(a^3 - 2a^2 - a - 6) = 0$

or $a(a - 3)(a^2 + a + 2) = 0$

$\therefore$ $a = 0$ or $a = 3$ or $a^2 + a + 2 = 0$

Similarly we can get, $b = 0$ or $b = 3$

or $b^2 + b + 2 = 0$ (By eliminating 'a')

So, $a = b = 0$ or $a = b = 3$

or a, b are roots of $x^2 + x + 2 = 0$.

32. (b, c) : Given equation can be expressed as

$\pi^e(x - \pi)(x - \pi - e) + e^\pi(x - e)(x - \pi - e) + (\pi^\pi + e^e)(x - e)(x - \pi) = 0$

Let $f(x) = \pi^e(x - \pi)(x - \pi - e) + e^\pi(x - e)(x - \pi - e) + (\pi^\pi + e^e)(x - e)(x - \pi)$

$f(e) = \pi^e(e - \pi)(-\pi) > 0$

and $f(\pi) = e^\pi(\pi - e)(-e) < 0$

Hence given equation has a real root in (e, π)

Again $f(\pi + e) = (\pi^{\pi} + e^{e})\pi \cdot e > 0$
$\because$ $\pi + e > \pi$, it concludes, it has a real root in $(\pi, \pi + e)$
Also $\because$ $\pi - e < \pi$
Hence, $f(x)$ has two real roots in $(\pi - e, \pi + e)$

(33-35) :
Let $x^4 + ax^3 + bx^2 + cx + d$
$= (x - x_1)(x - x_2)(x - x_3)(x - x_4)$
Let $(x - x_1)(x - x_2) = x^2 + px + q$
and $(x - x_3)(x - x_4) = x^2 + px + r$
$\therefore$ $q = x_1x_2$ and $r = x_3x_4$
$\therefore$ $x^4 + ax^3 + bx^2 + cx + d$
$= x^4 + 2px^3 + (p^2 + q + r)x^2 + p(q + r)x + qr$
$\therefore$ $a = 2p$, $b = p^2 + q + r$, $c = p(q + r)$, $d = qr$
Clearly, $a^3 - 4ab + 8c = 0$ ...(i)

33. (b) : If $a = 2 \Rightarrow b - c = 1$

34. (c) : Investigating the nature of the cubic equation of 'a'
Let $f(a) = a^3 - 4ab + 8c$
$f'(a) = 3a^2 - 4b$
If $b < 0 \Rightarrow f'(a) > 0$
$\therefore$ The equation $a^3 - 4ab + 8c = 0$ has only one real root.

35. (a) : Substituting $c = 1 - b$ in Eq. (i), we have
$(a + 2)[(a - 1)^2 + 3 - 4b] = 0 \Rightarrow 4b - 3 > 0$
$\Rightarrow b > \frac{3}{4} \Rightarrow c < \frac{1}{4}$

(36-38)
Let $\alpha, \beta, \gamma, \delta$ be roots of the given equation
$\alpha + \beta + \gamma + \delta = 12$ ...(i)
$\Sigma\alpha\beta = b$ ...(ii)
$\Sigma\alpha\beta\gamma = -c$ ...(iii)
$\alpha\beta\gamma\delta = 81$ ...(iv)
As A.M. $\geq$ G.M.
$\therefore \frac{\alpha+\beta+\gamma+\delta}{4} \geq (\alpha\beta\gamma\delta)^{1/4} \Rightarrow \frac{12}{4} \geq (81)^{1/4}$
But as A.M. = G.M.
$\therefore$ $\alpha = \beta = \gamma = \delta = 3$

36. (b) : $b = \Sigma\alpha\beta = 6 \times 9 = 54$

37. (b) : $c = \Sigma\alpha\beta\gamma = -(4 \times 27) = -108$

38. (c) : $2bx + c = 0 \Rightarrow 108x - 108 = 0 \Rightarrow x = 1$

39. (a) **40. (c)** **41. (b)**

(42-44) :

42. (d) : Let $x^4 + ax^3 + bx^2 + ax + 1 = (x^2 + kx + 1)^2$
Comparing $2k = a$; $b = k^2 + 2$
$\Rightarrow$ $a + b = (k + 1)^2 + 1$, it can be 4.

43. (a) : The two equal positive roots must be 1, 1, and let the negative roots be $\alpha, \frac{1}{\alpha}$ $(\alpha \neq -1)$
Now $-a = 2 + \alpha + \frac{1}{\alpha} \Rightarrow a = -\alpha - \frac{1}{\alpha} - 2 > 0$
$\therefore$ The least integral value is '1'.

44. (b) : Given equation is $t^2 + at - 3 = 0$ where $t = x + \frac{1}{x}$
both roots must lie between -2, 2
$\Rightarrow \frac{-1}{2} < a < \frac{1}{2} \Rightarrow a = 0$

(45-47) :

45. (c) : Solving it, we get
$\sin(\alpha + \beta) = \sin\alpha \pm \sqrt{-\cos^2\beta}$
$\Rightarrow \cos\beta = 0 \Rightarrow \beta = \frac{\pi}{2}, \frac{3\pi}{2}$
(i) If $\beta = \frac{\pi}{2} \Rightarrow \tan\alpha = 1$, $\alpha \in \left\{\frac{\pi}{4}, \frac{5\pi}{4}\right\}$
(ii) If $\beta = \frac{3\pi}{2} \Rightarrow \tan\alpha = -1$, $\alpha \in \left\{\frac{3\pi}{4}, \frac{7\pi}{4}\right\}$

46. (b) : We have $x + y + z = 6$ ...(i)
$xy + yz + zx = 7$ ...(ii)
From (i), $z = 6 - x - y$ and putting it in (ii),
we get $xy + y(6 - x - y) + x(6 - x - y) = 7$
or $y^2 + (x - 6)y + (x^2 - 6x + 7) = 0$
Since y is real, $(x - 6)^2 - 4(x^2 - 6x + 7) \geq 0$
$\Rightarrow 3x^2 - 12x - 8 \leq 0$
$\Rightarrow \frac{6-2\sqrt{15}}{3} \leq x \leq \frac{6+2\sqrt{15}}{3}$

47. (d) : $3^x\left(9 \cdot 3^x + \frac{1}{3^x} + (a^2 - 4a - 2)\right) > 0$
$\Rightarrow 3^x\left(\left(3 \cdot 3^{x/2} - \frac{1}{3^{x/2}}\right)^2 + (a - 2)^2\right) > 0$
$\Rightarrow a \in R - \{2\}$

48. A $\rightarrow$ q; B $\rightarrow$ r; C $\rightarrow$ q, r, s, t; D $\rightarrow$ p

49. A $\rightarrow$ t ; B $\rightarrow$ r ; C $\rightarrow$ p, q, r, s ; D $\rightarrow$ q
(A) A.M. $\geq$ G.M.
$\frac{2a+3b}{2} \geq \sqrt{2a \cdot 3b}$ or, $\frac{ab}{2} \geq \sqrt{6ab}$
(B) L.H.S. Min. = R.H.S. Max. when $x = 2$
(C) $f(x) = x^3 - 6x^2 + 9x + \lambda$
$f'(x) = 3x^2 - 12x + 9 = 3(x - 1)(x - 3)$
$f'(x) \leq 0 \ \forall \ x$ in $(1, 3)$
For $f(x) = 0$ to have exactly one root in $(1, 3)$
$f(1) f(3) < 0$.
$\lambda(\lambda + 4) < 0 \quad \therefore \ -4 < \lambda < 0$
$\Rightarrow [\lambda + 3] = -1, 0, 1, 2$
(D) Since $x = 1$ satisfies $(b - c)x^2 + (c - a)x + a - b = 0$
$\therefore$ $x = 1$ is common root of both equations
Therefore, $1 + 3\lambda + 2 = 0$

Therefore, $\lambda = -1$

50. A → r; B → p ; C → q ; D → s

(A) Given $a^2 - 4a + 1 = 4 \Rightarrow a^2 + 1 = 4(1 + a)$

$$y = \frac{(a-1)(a^2+1)}{a^2-1} = \frac{a^2+1}{a+1} = \frac{4(a+1)}{a+1} = 4$$

(B) $\sqrt[4]{|x-3|^{x+1}} = \sqrt[3]{|x-3|^{x-2}}$

Taking log on both sides

$$\frac{x+1}{4}\log|x-3| = \frac{x-2}{3}\log|x-3|$$

$$\Rightarrow \log|x-3| = 0 \text{ or } \frac{x+1}{4} - \frac{x-2}{3} = 0$$

$\Rightarrow x = 4, 2$ or $x = 11$

(C) Critical points, $x = 0, 6$

Case-I: $x \geq 6 \Rightarrow 3^x + 1 - (3^x - 1) = 2\log_5(x - 6)$

$\Rightarrow x = 11$

Case-II: $0 \leq x \leq 6 \Rightarrow 3^x + 1 - (3^x - 1) = 2\log_5(6 - x)$

$\Rightarrow x = 1$

Case-III: $x < 0 \Rightarrow 3^x + 1 + 3^x - 1 = 2\log_5(6 - x)$

$\Rightarrow 3^x = \log_5(6 - x) \Rightarrow$ no solution

(D) $$\frac{2n}{2}(4 + (2n-1)3) = \frac{n}{2}(114 + (n-1)2)$$

$\Rightarrow n = 11$

51. (3) : $|x - p| = x - p$ (since $x \geq p$)

$|x - 15| = 15 - x$ (since $x \leq 15$)

$|x - (p + 15)| = (p + 15) - x$ (as $15 + p > x$)

$\therefore$ expression reduces to

$E = x - p + 15 - x + p + 15 - x$

$E = 30 - x$

$\therefore$ E_{min} occurs when $x = 15$

$\therefore \lambda = 15$

52. (4) : Since $P(x)$ divides both given polynomial

Hence $P(x)$ also divides

$(3x^4 + 4x^2 + 28x + 5) - 3(x^4 + 6x^2 + 25)$

$= -14x^2 + 28x - 70 = -14(x^2 - 2x + 5)$

Which is a quadratic, Hence $P(x) = x^2 - 2x + 5$

$\therefore P(1) = 4$

53. (1) : Let α be the root of the given cubic where α can take values a, b, c

Hence $\alpha^3 + \alpha^2 - 333\alpha - 1002 = 0$

or $\alpha^3 = 1002 + 333\alpha - \alpha^2$

$\therefore \Sigma\alpha^3 = \Sigma 1002 + 333\Sigma\alpha - \Sigma\alpha^2 = 3006 + 333\Sigma\alpha - [(\Sigma\alpha)^2 - 2\Sigma\alpha_1\alpha_2]$

But $\Sigma\alpha = -1$, $\Sigma\alpha_1\alpha_2 = -333$

$\therefore a^3 + b^3 + c^3 = 3006 - 333 - [1 + 666]$

$= 3006 - 333 - 667 = 3006 - 1000 = 2006 = P$

54. (3) : $(x + 1)^5 = 2(x^5 + 1)$

Let $f(x) = \dfrac{(x+1)^5}{(x^5+1)}$ $(x \neq -1)$

$$\Rightarrow f'(x) = \frac{5(x+1)^4(1-x^4)}{(x^5+1)^2}$$

$\Rightarrow x = 1$ is maximum.

As, $f(0) = 1$ and $f(1) = 16$

And $\lim\limits_{x\to\pm\infty} f(x) = 1 \Rightarrow f(x) = 2$ has two solutions but given equation has three solutions because $x = -1$ included.

55. (8) : On putting $\log_3 x = t$, we get

$2t^2 - |t| + a = 0$ ...(i)

If $t > 0$, then $2t^2 - t + a = 0$ ...(ii)

If $t < 0$, then $2t^2 + t + a = 0$ ...(iii)

If eq. (i) has four roots then eq. (ii) must have both roots positive and eq. (iii) has both roots negative. Now, eq. (ii) has both roots positive, if $D > 0$.

and $a/2 > 0$

$\Rightarrow 1 - 8a > 0, a > 0$

$$\Rightarrow a \in \left(0, \frac{1}{8}\right)$$

Again, eq. (iii) has both roots negative, if $D > 0$, $a/2 > 0$.

We again get, $a \in \left(0, \dfrac{1}{8}\right) \Rightarrow K = 8$

On solving +ve and –ve sign, we get $a \geq \dfrac{3}{4}$

$\therefore m + n = 7$

56. (3) **57. (7)**

58. (4) : Suppose m is a positive integer such that $n^2 + 96 = m^2$ then $(m - n)(m + n) = 96$

As $m - n < m + n$ and $m - n$, $m + n$ both must be even.

So, the only possibilities are

$m - n = 2$, $m + n = 48$: $m - n = 4$, $m + n = 24$

$m - n = 6$, $m + n = 16$: $m - n = 8$, $m + n = 12$

So, the solutions of (m, n) are (25, 23), (14, 10), (11, 5), (10, 2).

59. (1) : Here, $\alpha + \beta = -p = \gamma + \delta$

$(\alpha - \gamma)(\alpha - \delta) = \alpha^2 - \alpha(\gamma + \delta) + \gamma\delta = \alpha^2 - \alpha(\alpha + \beta) + r$

$= -\alpha\beta + r = q + r$

Similarly $(\beta - \gamma)(\beta - \delta) = q + r$

So, ratio is 1.

60. (7) : Let us form that equation having roots

$y = g(x_i)$ *i.e.*, $y = x^2 - 2$

$$\Rightarrow x = \sqrt{y+2} \Rightarrow (\sqrt{y+2})^5 + (\sqrt{y+2})^2 + 1 = 0$$

$\Rightarrow y^5 + 10y^4 + 40y^3 + 79y^2 + 74y + 23 = 0$

$\therefore g(x_1) \ldots g(x_5) =$ Product of roots $= -23$

$x_1x_2x_3x_4x_5 = -1 \Rightarrow g(x_1x_2 \ldots x_5) = -1$

$\Rightarrow g(x_1)g(x_2) \ldots g(x_5) - 30g(x_1x_2 \ldots x_5) = 7$

MATHS MUSING

SOLUTION SET-150

1. (d) : The number of terms till last n is $1 + 2 + 3 + \ldots$

$\ldots + n = \dfrac{n(n+1)}{2} = 2016$ for $n = 63$

The sum of these 2016 terms is $1^2 + 2^2 + 3^2 + \ldots + 63^2 = 85344$.

S = The sum of first 2015 terms = 85344 – 63 = 85281, with digit sum 24.

2. (b) : Circle through $(0, a)$ and $(0, -a)$ is $(x - \lambda)^2 + y^2 = \lambda^2 + a^2$.

Centre $(\lambda, 0)$, radius $r = \sqrt{\lambda^2 + a^2}$.

It touches $y = mx + c$

$\Rightarrow (m\lambda + c)^2 = (m^2 + 1)(\lambda^2 + a^2)$

$\lambda^2 - 2mc\lambda + a^2(m^2 + 1) - c^2 = 0$

Roots $\lambda_1, \lambda_2 \Rightarrow \lambda_1\lambda_2 = a^2(m^2 + 1) - c^2$ ...(i)

S_1, S_2 are orthogonal $\Rightarrow (\lambda_1 - \lambda_2)^2 = \lambda_1^2 + \lambda_2^2 + 2a^2$

$\Rightarrow \lambda_1\lambda_2 = -a^2$. Now (i) $\Rightarrow a^2(m^2 + 2) = c^2$.

3. (c) : $\overrightarrow{OA} = \vec{a}, \overrightarrow{OB} = \vec{b}, \overrightarrow{OC} = \vec{c}, |\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$\vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{c} = \vec{c}\cdot\vec{a} = \dfrac{1}{2}$

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix} = \frac{1}{2}, [\vec{a}\ \vec{b}\ \vec{c}] = \frac{1}{\sqrt{2}}$$

$\vec{r} = \lambda\vec{a}, \vec{r} = \vec{b} + m(\vec{c} - \vec{b})$

$$\text{S.D.} = \vec{b}\cdot\frac{(\vec{a}\times(\vec{c}-\vec{b}))}{|\vec{a}\times(\vec{c}-\vec{b})|} = \frac{[\vec{a}\,\vec{b}\,\vec{c}]}{1} = \frac{1}{\sqrt{2}}$$

4. (c) : $$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3 - 3\lambda^2 + \lambda + 1 = 0$$

$\therefore\ A^2 - 3A + I + A^{-1} = 0 \Rightarrow A^{-1} + A^2 = 3A - I.$

5. (b) : $(1, 1)$ and $\left(\dfrac{1}{\sqrt{2}}, \dfrac{\sqrt{5}}{2}\right)$ form $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

$$\Rightarrow \frac{1}{a^2} + \frac{1}{b^2} = 1, \frac{1}{2a^2} + \frac{5}{4b^2} = 1 \Rightarrow a = \sqrt{3}, b = \sqrt{\frac{3}{2}}$$

$$e = \sqrt{1 - \frac{b^2}{a^2}} = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$$

6. (b,c) : $i = \text{cis}\left(2n + \dfrac{1}{2}\right)\pi = e^{\left(2n+\frac{1}{2}\right)\pi i}$

$\therefore\ i^i = e^{-\left(2n+\frac{1}{2}\right)\pi} = e^{-\pi/2}, e^{3\pi/2}$ for $n = 0, -1$

7. (c) : $\sin^{-1}x = x + \dfrac{1}{2}\dfrac{x^3}{3} + \dfrac{1}{2}\cdot\dfrac{3}{4}\cdot\dfrac{x^5}{5} + \ldots$

$$I = \int_0^{\pi/2} \sin^{-1}(c\cos x)\,dx$$

$$= \int_0^{\pi/2} \left(c\cos x + \frac{1}{2}\frac{c^3\cos^3 x}{3} + \ldots\right)dx = \frac{c}{1^2} + \frac{c^3}{3^2} + \frac{c^5}{5^2} + \ldots$$

$c = 1$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2} = \int_0^{\pi/2}\sin^{-1}(\cos x)\,dx$$

$$= \int_0^{\pi/2}\left(\frac{\pi}{2} - x\right)dx = \frac{\pi^2}{8}$$

$$S = \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{8} + \frac{S}{4} \Rightarrow S = \frac{\pi^2}{6}$$

8. (a) : $S = \dfrac{1}{1^2} - \dfrac{1}{2^2} + \dfrac{1}{3^2} - \dfrac{1}{4^2} + \ldots$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{4}\left(\frac{1}{1^2} + \frac{1}{2^2} + \ldots\right) = \frac{\pi^2}{8} - \frac{\pi^2}{24} = \frac{\pi^2}{12}$$

9. (9) : Heads cannot occur on consecutive tosses if the number of heads is 0 to 5.

$$\text{Prob.} = \frac{1}{2^{10}}\sum_{r=0}^{5}\binom{10-r+1}{r}$$

$$= \frac{1}{2^{10}}(1 + 10 + 36 + 56 + 35 + 6) = \frac{9}{64}, m = 9$$

10. (d) : $(1 + x + x^2)^n$

$$= 1 + \binom{n}{1}x(1+x) + \binom{n}{2}x^2(1+x)^2 + \binom{n}{3}x^3(1+x)^3 + \ldots$$

$$\therefore\ C_n = \binom{n}{n} + \binom{n}{n-1}\binom{n-1}{1} + \binom{n}{n-2}\binom{n-2}{2} + \ldots$$

$$= 1 + \frac{n(n-1)}{(1!)^2} + \frac{n(n-1)(n-2)(n-3)}{(2!)^2} + \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{(3!)^2} + \ldots$$

$C_3 = 1 + 3\cdot2 = 7$

$C_4 = 1 + 4\cdot3 + 3\cdot2 = 19$

$$C_5 = 1 + 5\cdot4 + \frac{5\cdot4\cdot3\cdot2}{4} = 51$$

$$C_7 = 1 + 7\cdot6 + \frac{7\cdot6\cdot5\cdot4}{4} + \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2}{36}$$

$= 1 + 42 + 210 + 140 = 393.$

Solution Sender of Maths Musing

SET-150

1. Khokon Kumar Nandi (W.B.)

SET-149

1. Swarnendu Bhattacharjee (W.B.)

concept BOOSTERS

Class XII

Functions

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This column is aimed at Class XII students so that they can prepare for competitive exams such as JEE Main/Advanced, etc. and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

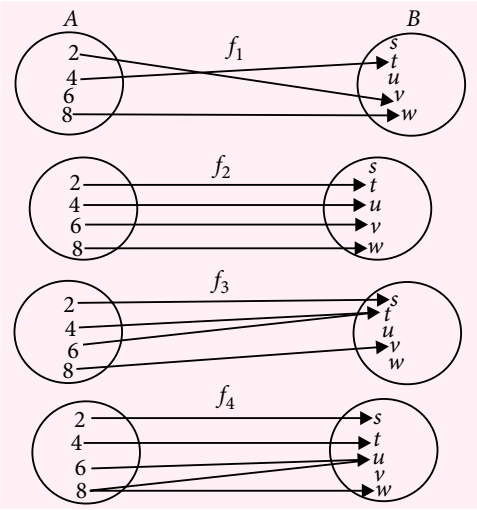
DEFINITION OF FUNCTION

Let A and B be two non–empty sets. Then a function 'f' from set A to set B is a rule which associates elements of set A to elements of set B such that an element of set A is associated to a unique element in set B.

All elements of set A are associated to element l in set B.

Terms such as "map" (or mapping), "correspondence" are used as synonyms for function. If f is a function from a set A to set B, then we write $f : A \to B$ or $A \xrightarrow{f} B$, which is read as f is a function from A to B or f maps A to B.

Example : Let $A = \{2, 4, 6, 8\}$ and $B = \{s, t, u, v, w\}$ be two sets and let f_1, f_2, f_3 and f_4 be rules associating elements of A to elements of B as shown in the following figures.



Now see that f_1 is not a function from set A to set B, since there is an element $6 \in A$ which is not associated to any element of B, but f_2 and f_3 are the functions from A to B, because under f_2 and f_3 each element in A is associated to a unique element in B. But f_4 is not a function from A to B because an element $8 \in A$ is associated to two elements u and w in B.

Domain : Set A is called domain of f *i.e.* Set of those elements from which functions is to be defined.

Co–domain : Here set B is called co–domain of function.

Range : Set of images of each element in A, is called range of f.

Note : Range $\subseteq$ Co-domain

Real Valued Function : All those functions of which domain and co-domain are subsets of R are called real valued functions. In this case for a given function we have to find domain and range.

Bounded Function : A function 'f' is said to be bounded if $|f(x)| \le m$, for some finite 'm' for every x in domain of f.

Equality of two functions : Two functions f and g are said to be equal functions, if and only if

(i) domain of f = domain of g

(ii) co–domain of f = co–domain of g

(iii) $f(x) = g(x)\ \forall\ x \in$ their domain

Example :

If $f : A \to B$, $A = \{1, 2\}$, $B = \{10, 13\}$, $f(x) = x^2 + 9$ and $g : A \to B$, $g(x) = 3x + 7$, then $f = g$ because domain and co-domain of both f and g are same. Also $f(1) = 10 = g(1)$; $f(2) = 13 = g(2)$

ALGEBRA OF FUNCTIONS

If $f : D_1 \to R$ and $g : D_2 \to R$, then, we describe functions $f + g$, $f - g$, fg and f/g as follows

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$f + g : D \to R$ is a function defined by

$(f + g)(x) = f(x) + g(x)$

$f - g : D \to R$ is a function defined by

$(f - g)(x) = f(x) - g(x)$

$fg : D \to R$ is a function defined by

$(fg)(x) = f(x)\, g(x)$

$\frac{f}{g} : C \to R$ is a function defined by

$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0,$

where $D = D_1 \cap D_2$ and $C = \{x \in D : g(x) \neq 0\}$

COMPOSITE FUNCTION

Consider two functions $f : X \to Y$ and $g : Y \to Z$. one can define $h : X \to Z$ such that

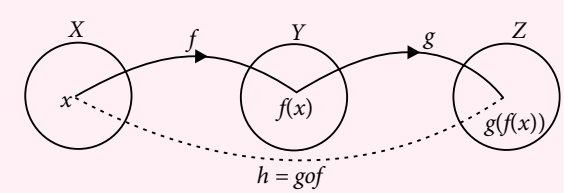
$h(x) = g\{f(x)\}$

Domain of $gof(x)$ *i.e.* $g\{f(x)\}$

$= \{x : x \in \text{Dom } f, f(x) \in \text{Dom } g\}$

Domain of $fog(x)$ *i.e* $f\{g(x)\}$

$= \{x : x \in \text{Dom } g, g(x) \in \text{Dom } f\}$



EVEN AND ODD FUNCTIONS :

If $f : X \to Y$ is a real valued function such that for all $x \in D \Rightarrow -x \in D$ (where D = domain of f) and if $f(-x) = f(x)$ for every $x \in D$ then f is said to be an even function and if $f(-x) = -f(x)$ then f is said to be an odd function. Even functions are symmetric about the y–axis (*i.e.* if (x, y) lies on the curve, then $(-x, y)$ also lies on the curve, and odd functions are symmetric about the origin (*i.e.* if (x, y) lies on the curve, then $(-x, -y)$ also lies on the curve.

Remarks

Every function defined in symmetric interval D (*i.e.* $x \in D \Rightarrow -x \in D$) can be expressed as a sum of an even and an odd function.

$$f(x) = \left(\frac{f(x) + f(-x)}{2}\right) + \left(\frac{f(x) - f(-x)}{2}\right)$$

Let $h(x) = \left(\frac{f(x) + f(-x)}{2}\right)$

and $g(x) = \left(\frac{f(x) - f(-x)}{2}\right)$

It can now easily be shown that $h(x)$ is even and $g(x)$ is odd.

The first derivative of an even function is an odd function and vice – versa.

If $x = 0$, $x \in$ Domain of f, then for odd function $f(x)$, which is continuous at $x = 0$, $f(0) = 0$, *i.e.* if for a function $f(0) \neq 0$, then that function cannot be odd. It follows that for a differentiable even function $f'(0) = 0$ *i.e.* if for a differentiable function $f'(0) \neq 0$ then the function cannot be even.

PERIODIC FUNCTION

Definition : A function $f(x)$ is said to be periodic function if, there exists a fixed positive real number T independent of x, such that $f(x + T) = f(x)$ $\forall\ x \in$ Domain & $x + T \in$ domain.

T is called one of the period of the function.

In other words , a function is said to be periodic function if its each value is repeated after a definite interval.

Here the least positive value of T (independent of x) is called the fundamental period of the function.

Clearly, $f(x) = f(x + T) = f(x + 2T) = f(x + 3T) = \ldots$

For example :

- $\sin x$, $\cos x$, $\sec x$ and $\operatorname{cosec} x$ are periodic functions with period 2π.
- $\tan x$ and $\cot x$ are periodic functions with period π.
- $|\sin x|$, $|\cos x|$, $|\tan x|$, $|\cot x|$, $|\sec x|$, $|\operatorname{cosec} x|$ are periodic functions with period π.
- $\sin^n x$, $\cos^n x$, $\sec^n x$, $\operatorname{cosec}^n x$ are periodic functions with period 2π or π according as n is odd or even.
- $\tan^n x$ and $\cot^n x$ are periodic functions with period π whether n is odd or even.

Properties of Periodic Function :

If $f(x)$ is periodic with period T, then

$f(x + c)$ is periodic with period T.

$f(x) \pm c$ is periodic with period T.

$f(ax + b)$ has period $\frac{T}{|a|}$, *i.e.*, period is affected only by coefficient of x ; where a, b, c are constants with $a,\ b \neq 0$.

If $f(x)$ and $g(x)$ be two periodic functions with period p & q respectively, then their any combination will be periodic function with one period equal to L.C.M of p & q provided L.C.M of p & q exists.

Note : All periodic functions can be analyzed over an interval of one period within the domain as the same pattern shall be repetitive over the entire domain.

CLASSIFICATION OF FUNCTIONS

(i) One–One Function (Injective) :

If each element in the domain of a function has a distinct image in the co–domain, then the function is said to be one–one function and is also known as Injective Function.

e.g. $f : R \to R^+$ given by $y = e^x$

$g : R \to R$, $g(x) = 3x - 7$ are one – one functions.

or, $f : A \to B$ is one – one

$\Leftrightarrow a \neq b \Rightarrow f(a) \neq f(b)$ for all $a, b \in A$

$\Leftrightarrow f(a) = f(b) \Rightarrow a = b$ for all a, $b \in A$

(ii) Many–One Function :

If there are two or more than two elements of domain having the same image then $f(x)$ is called many – one function.

e.g. $f : R \to R^+$ given by $f(x) = x^2 + 4$

$g : R \to R^+$ given by $g(x) = x^8 + x^4 + x^2 + 4$

Both functions are many one.

If the graph of $y = f(x)$ is given and a line parallel to x–axis cuts the curve at more than one point then the function is many–one.

or, $f : A \to B$ is a many – one function if there exist $x, y \in A$ such that $x \neq y$ but $f(x) = f(y)$.

e.g. $y = \sin x$, $y = \cos x$, $y = \tan x$, $y = x^2$, $y = x^4$, … are many–one functions.

(iii) Onto Function (Surjective) :

Let $f : X \to Y$ be a function. If each element in the co–domain Y has at least one pre–image in the domain X *i.e.* Range f = Co-domain, then f is called onto.

Onto function is also called surjective and if function be both one–one and onto then function is called Bijective.

or, $f : A \to B$ is a surjection iff for each $b \in B$ $\exists$ $a \in A$ such that $f(a) = b$.

e.g. If $f : R^+ \to R$ is defined by $y = \log_2 x$, then $f(x)$ is onto function.

(iv) Into Function :

If there exist one or more than one element in the co–domain Y which is not an image of any element in the domain X. Then f is into.

In other words $f : A \to B$ is an into function if it is not an onto function.

e.g. Let $f : R \to R$ is defined by $y = x^2 + 1$, then $f(x)$ is an into function. But when $f : R \to R^+$ is defined by $y = x^2 + 1$, then $f(x)$ is not into function.

Note :

- If domain of $f(x)$ is continuous and $\frac{dy}{dx} > 0, \forall x$ in domain then f is one–one, where equality exist at discrete point.
- If domain of $f(x)$ is continuous and $\frac{dy}{dx} < 0, \forall x$ in domain then f is one–one, where equality exist at discrete point.
- If a continuous function $f(x)$ which has either local minima or local maxima or both then $f(x)$ will be many–one.
- Every even function is many–one.
- Every periodic function is many–one.

INVERSE FUNCTION

If $f : X \to Y$ be a function defined by $y = f(x)$ such that f is both one–one and onto, then there exists a unique function $g : Y \to X$ such that for each $y \in Y$, $g(y) = x$. The function g so defined is called the inverse of f and denoted by f^{-1}. Also, f is the inverse of g and the two functions f and g are said to be inverse of each other.

$f(f^{-1}(x)) = x, \forall x \in Y$ and $f^{-1}(f(x)) = x, \forall x \in X$

Note that f and f^{-1} are symmetric about the line $y = x$.

METHOD OF FINDING INVERSE OF A FUNCTION :

1. If you are asked to check whether the given function $y = f(x)$ is invertible, you need to check that $y = f(x)$ is one–one and onto.
2. If you are asked to find the inverse of a bijective function $f(x)$, you do the following : if f^{-1} be the inverse of 'f', then $f(f^{-1}(x)) = x$. Apply the formula of f on $f^{-1}(x)$ and use of the above identity to solve for $f^{-1}(x)$.

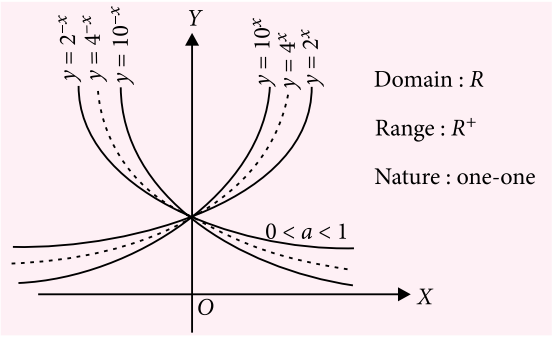
Some standard functions given below along with their inverse functions

	Function	Inverse Function
(i)	$f : [0, \infty) \to [0, \infty)$ defined by $f(x) = x^2$	$f^{-1} : [0, \infty) \to [0, \infty)$ defined by $f^{-1}(x) = \sqrt{x}$
(ii)	$f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ defined by $f(x) = \sin x$	$f^{-1} : [-1, 1] \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ defined by $f^{-1}(x) = \sin^{-1} x$
(iii)	$f : [0, \pi] \to [-1, 1]$ defined by $f(x) = \cos x$	$f^{-1} : [-1, 1] \to [0, \pi]$ defined by $f^{-1}(x) = \cos^{-1} x$

SOME ELEMENTARY FUNCTIONS

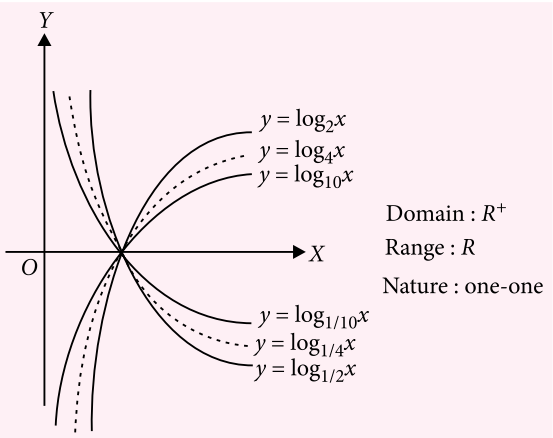
- **General Exponential Function**

If $a > 0$, $a \neq 1$ then the function defined by $f(x) = a^x$, $x \in R$ is called an Exponential Function with base a.



- **Logarithmic Function**

If $a > 0$, $a \neq 1$, then the function $y = \log_a x$, $x \in R^+$(set of positive real numbers) is called the Logarithmic Function with base a.



- **Polynomial Function :**

If a function is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \ldots\ldots\ldots + a_{n-1} x + a_n$, where n is a non negative integer and $a_0, a_1, a_2, \ldots\ldots\ldots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree n.

Note :

(A) A polynomial of degree one with no constant term is called an odd linear function. *i.e.* $f(x) = ax$, $a \neq 0$.

(B) There are two polynomial functions satisfying the relation $f(x).\ f(1/x) = f(x) + f(1/x)$.

They are

(i) $f(x) = x^n + 1$

(ii) $f(x) = 1 - x^n$, where n is a positive integer.

(C) $f(x) = c$ and $c \neq 0$ is a polynomial of degree zero.

(D) $f(x) = 0$ is a polynomial but degree not defined.

Note : Functions given in (C) and (D) are also called constant functions.

- **Algebraic Function :**

y is an algebraic function of x, if it satisfies an algebraic equation of the form,

$P_0(x)y^n + P_1(x)y^{n-1} + \ldots\ldots + P_{n-1}(x)y + P_n(x) = 0$,

where n is a positive integer and $P_0(x)$, $P_1(x)$, $\ldots\ldots P_n(x)$ are polynomials in x. e.g. $y = |x|$ is an algebraic function, since it satisfies the equation $y^2 - x^2 = 0$.

Note that all polynomial functions are Algebraic but not the converse. A function that is not algebraic is called a Transcendental Function.

- **Rational Function**

The function which can be written as the quotient of two polynomial functions is said to be a rational function.

If $P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n$

and $Q(x) = b_0 + b_1x + b_2x^2 + \ldots + b_mx^m$

be two polynomial functions then a function f defined by $f(x) = \dfrac{P(x)}{Q(x)}$; $Q(x) \neq 0$ is a rational function of x.

Example : $f(x) = \dfrac{7x^4 - x^2 + 2}{x^2 - 4x + 3}$ is a rational function which is defined for all real values of x except 1 and 3.

- **Identity Function**

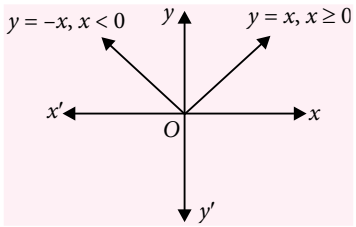
A map $f : R \to R$ is said to be an identity function, iff $f(x) = x$, $\forall\ x \in R$.

Domain of identity function = R

Range of identity function = R

- **Modulus Function**

$$f(x) = |x| = \begin{cases} -x, & x < 0 \\ x, & x \geq 0 \end{cases}$$



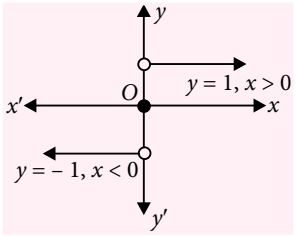
Domain : R, Range : $[0, \infty)$

It is an even continuous and many – one function.

Graph is symmetrical with respect to y–axis.

- **Signum Function**

$$f(x) = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$



Domain : R, Range : $\{-1, 0, 1\}$. It is a many – one and discontinuous function.

- **Greatest Integer Function**

If $f(x) = k \; \forall \; x \in [k, k + 1)$, where k is any integer, then f is called greatest integer function usually denoted by $f(x) = [x]$

Examples : $[3.7] = 3$, $[-3.2] = -4$, $[5] = 5$ etc.

Properties of Greatest Integer Function

1. $x - 1 < [x] \le x$
2. $[x] + [-x] = \begin{cases} 0; x \in I \\ -1; x \notin I \end{cases}$
3. $[x] = n \implies n \le x < n + 1$
   $[x] \ge n \implies x \ge n, n \in I$
   $[x] \le n \implies x < (n + 1), n \in I$
4. $[x + n] = [x] + n$, where $n \in I$

- **Fractional Part of x**

$f(x) = x - [x], x \in R$

$$i.e., f(x) = \{x\} = \begin{cases} x+1, & x \in [-1,0) \\ x, & x \in [0,1) \\ x-1, & x \in [1,2) \end{cases}$$

Domain : R, Range : $[0, 1)$, Nature : Many–one

This is a periodic function with period 1. It is discontinuous at all integers.

Properties of Fractional Part of x

1. $x = [x] + \{x\}$, where $[.]$ and $\{.\}$ denotes the integral and fractional part of x respectively.
2. $\{x\} + \{-x\} = \begin{cases} 0 ; x \in I \\ 1 ; x \notin I \end{cases}$

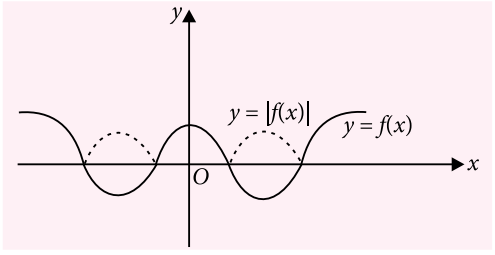
ELEMENTARY TRANSFORMATIONS OF GRAPHS

(i) Drawing the graph of $y = |f(x)|$ from the known graph of $y = f(x)$

$$|f(x)| = \begin{cases} f(x), \text{ if } f(x) \ge 0 \\ -f(x), \text{ if } f(x) < 0 \end{cases}$$

It means that the graph of $f(x)$ and $|f(x)|$ would coincide if $f(x) \ge 0$ and the portions where $f(x) < 0$ would get inverted in the upward direction.

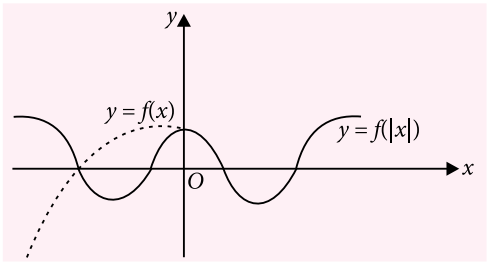
The above figure would make the procedure clear.



(ii) Drawing the graph of $y = f(|x|)$ from the known graph of $y = f(x)$

$$f(|x|) = \begin{cases} f(x), & \forall x \ge 0 \\ f(-x), & \forall x < 0 \end{cases}$$

Thus $f(|x|)$ would be an even function. Graphs of $f(|x|)$ and $f(x)$ would be identical in the first and the fourth quadrants (as $x \ge 0$) and as such the graph of $f(|x|)$ would be symmetrical about the y–axis (as $(|x|)$ is even).

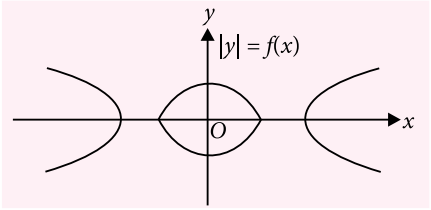


The figure would make the procedure clear.

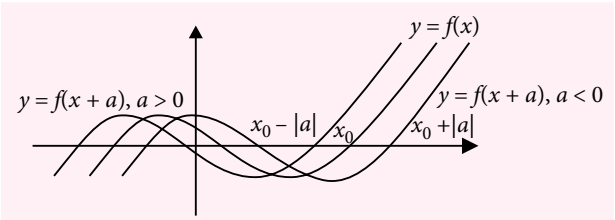
(iii) Drawing the graph of $|y| = f(x)$ from the known graph of $y = f(x)$

Clearly $|y| \ge 0$. If $f(x) < 0$, graph of $|y| = f(x)$ would not exist. And if $f(x) \ge 0$, $|y| = f(x)$ would give $y = f(x)$. Hence graph of $|y| = f(x)$ would exist only in the regions where $f(x) \ge 0$ and will be reflected about x–axis only in those regions. Regions where $f(x) < 0$ will be neglected.

Full lines show the graph of $|y| = f(x)$ and dotted lines depict the corresponding graph of $y = f(x)$.



(iv) Drawing the graph of $y = f(x + a)$, $a \in R$ from the known graph of $y = f(x)$

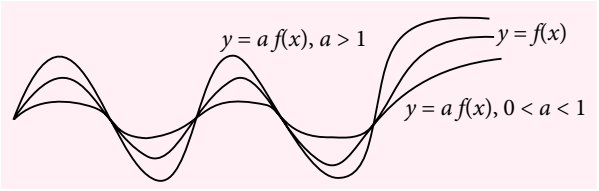


Let us take any point $x_0 \in$ domain of $f(x)$, and set $x + a = x_0$ or $x = x_0 - a$, $a > 0$

$\implies x < x_0$, and $a < 0 \implies x > x_0$. That mean x_0 and $x_0 - a$ would give us same abscissa for $f(x)$ and $f(x + a)$ respectively. As such for $a > 0$, graph of $f(x + a)$ can be obtained simply by translating the

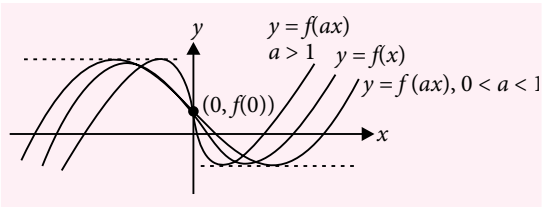
graph of $f(x)$ in the negative x–direction through a distance 'a' units. If $a < 0$, graph of $f(x + a)$ can be obtained by translating the graph of $f(x)$ in the positive x–direction through a distance a units. Accordingly, the graph of $f(x) + b$ can be obtained by translating the graph of $f(x)$ either in the positive y–axis direction (if $b > 0$) or in the negative y–axis direction (if $b < 0$), through a distance $|b|$ units.

(v) Drawing the graph of $y = a\,f(x)$ from the known graph of $y = f(x)$



It is clear that the corresponding points (points with same x-coordinates) would have their ordinates in the ratio of $1 : a$.

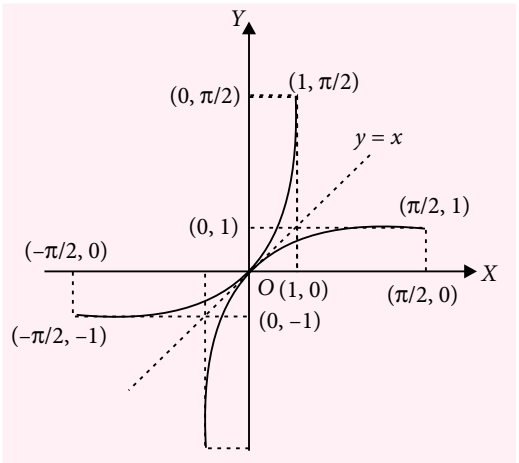
(vi) Drawing the graph of $y = f(ax)$ from the known graph of $y = f(x)$



Let us take any point $x_0 \in$ domain of $f(x)$. Let $ax = x_0$ or $x = \dfrac{x_0}{a}$

Clearly if $0 < a < 1$ then $x > x_0$ and $f(x)$ will stretch by $1/a$ units against y–axis, and if $a > 1$, $x < x_0$, then $f(x)$ will compress by 'a' units against y–axis.

(vii) Drawing the graph of $y = f^{-1}(x)$ from the known graph of $y = f(x)$



For drawing the graph of $y = f^{-1}(x)$ we have to first of all find the interval in which the function is bijective (invertible). Then take the reflection of $y = f(x)$ (within the invertible region) about the line $y = x$.

The reflected part would give us the graph of $y = f^{-1}(x)$.

e.g. let us draw the graph as of $y = \sin^{-1} x$. We know that $y = f(x) = \sin x$ is invertible if $f : \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right] \to [-1, 1]$, the inverse mapping would be $f^{-1} : [-1, 1] \to \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

SECTION-I

Single Correct Answer Type

1. Range of the function $y = \dfrac{2^x - 2^{-x}}{2^x + 2^{-x}}$ is

(a) R (b) $(-1, 1)$ (c) $[-1, 1]$ (d) $(0, 1)$

2. The range of the function

$f(x) = \cos^2 \dfrac{x}{4} + \sin \dfrac{x}{4},\ x \in R$ is

(a) $\left[0, \dfrac{5}{4}\right]$ (b) $\left[1, \dfrac{5}{4}\right]$ (c) $\left(-1, \dfrac{5}{4}\right)$ (d) $\left[-1, \dfrac{5}{4}\right]$

3. Let $f(x) = \dfrac{x - [x]}{1 + x - [x]},\ x \in R$, where $[\cdot]$ denotes the greatest integer function. Then, the range of f is

(a) $(0, 1)$ (b) $\left[0, \dfrac{1}{2}\right)$ (c) $[0, 1]$ (d) $\left[0, \dfrac{1}{2}\right]$

4. Let $f(x) = [x]\cos\left(\dfrac{\pi}{[x+2]}\right)$ where, $[\cdot]$ denotes the greatest integer function. Then, the domain of f is

(a) $x \in R$, x is not an integer

(b) $(-\infty, -2) \cup [-1, \infty)$

(c) $x \in R$, $x \neq -2$ (d) $(-\infty, -1]$

5. Range of the function $f(x) = x^2 + \dfrac{1}{x^2 + 1}$, is

(a) $[1, \infty)$ (b) $[2, \infty)$ (c) $\left[\dfrac{3}{2}, \infty\right)$ (d) $(-\infty, \infty)$

6. The inverse of $f(x) = (5 - (x - 8)^5)^{1/3}$ is

(a) $5 - (x - 8)^5$ (b) $8 + (5 - x^3)^{1/5}$

(c) $8 - (5 - x^3)^{1/5}$ (d) $(5 - (x - 8)^{1/5})^3$

7. The range of the function

$f(x) = |x - 1| + |x - 2| + |x + 1| + |x + 2|$

where, $x \in [-2, 2]$ is

(a) $[6, 8]$ (b) $[2, 4]$ (c) $[0, 4]$ (d) $\{1, 2\}$

8. Range of the function $f(x)=\frac{x^2+x+2}{x^2+x+1}; x\in R$ is

(a) $(1,\infty)$ (b) $\left(1,\frac{11}{7}\right]$ (c) $\left(1,\frac{7}{3}\right]$ (d) $\left[1,\frac{7}{5}\right]$

9. The range of $f(x) = -x^3 + x^2 - x + \cos^{-1}x$, is

(a) $[-1, 3+\pi]$ (b) $[0, \pi-1]$

(c) $[-1, 2+\pi]$ (d) $[-1, \pi]$

10. The range of the function

$f(x)=\log_{10}\left\{\sin^{-1}(\sqrt{x-5})+\frac{3\pi}{2}\right\}$ is

(a) $\left[\log\frac{3\pi}{2},\log 2\pi\right]$ (b) $\left[\log\frac{3\pi}{2},\log 3\pi\right]$

(c) $\left[\log\frac{3\pi}{2},\log \pi\right]$ (d) $\left[\log\frac{3\pi}{4},\log 2\pi\right]$

11. The domain of

$f(x)=\sqrt{x-2-2\sqrt{x-3}}-\sqrt{x-2+2\sqrt{x-3}}$, is

(a) $[3, 5]$ (b) $(3, 5)$ (c) $[5, \infty)$ (d) $[3, \infty)$

12. Minimum value of the function

$f(x) = x^3 (x^3 + 1)(x^3 + 2)(x^3 + 3), x \in R$, is

(a) -2 (b) -1

(c) 1 (d) none of these

13. The domain of the function

$f(x) = \log_{[x]} \{\text{sgn } (x^2)\}$ (where [.] is G.I.F) is

(a) $[2, \infty)$ (b) $(-2, 2)$

(c) $(-\infty, 2)$ (d) None of these

14. The domain of the function

$f(x) = \log_{10}\{1 - \log_{10} (x^2 - 5x + 10)\}$ is

(a) $(0, \infty)$ (b) $(0, 5)$

(c) $(-\infty, 0)$ (d) None of these

15. The range of the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 1$ is

(a) $[31, \infty)$ (b) $[-31, \infty)$

(c) $[3, \infty)$ (d) $[-3, \infty)$

16. The range of the function

$f(x)=\cos^{-1}\sqrt{\log_{[x]}\frac{|x|}{x}}$, where [.] = G.I.F is

(a) $\left\{\frac{\pi}{2}\right\}$ (b) $\{0\}$ (c) $\{\pi\}$ (d) $\{2\pi\}$

17. If $f(x)=x^2+x+\frac{3}{4}$ and $g(x) = x^2 + ax + 1$ be two real functions, then the range of a for which $g(f(x)) = 0$ has no real solution is

(a) $(-\infty, -2)$ (b) $(-2, 2)$

(c) $(-2, \infty)$ (d) $(2, \infty)$

18. If $f(x)$ is a polynomial function such that $|f(x)| \le 1\ \forall\ x \in R$ and $g(x)=\frac{e^{f(x)}-e^{|f(x)|}}{e^{f(x)}+e^{|f(x)|}}$, then the range of $g(x)$ is

(a) $[0, 1]$ (b) $\left[0,\frac{e^2+1}{e^2-1}\right]$

(c) $\left[0,\frac{e^2-1}{e^2+1}\right]$ (d) $\left[\frac{1-e^2}{1+e^2},0\right]$

19. The domain of definition of the function, $f(x)$ given by the equation $2^x + 2^y = 2$ is

(a) $0 < x \le 1$ (b) $0 \le x \le 1$

(c) $-\infty < x \le 0$ (d) $-\infty < x < 1$

20. If $e^x + e^{f(x)} = e$, then for $f(x)$

(a) domain = $(-\infty, 1)$, range = $(-\infty, 1)$

(b) domain = $(-\infty, -1]$, range = $(-\infty, 1]$

(c) domain = $(-\infty, 0]$, range = $(-\infty, 1]$

(d) domain = $(-\infty, 0]$, range = $(-\infty, 3]$

21. The range of the function $y = [x^2] - [x]^2, x \in [0, 2]$ where [.] denotes the integral part, is

(a) $\{0\}$ (b) $\{0, 1\}$ (c) $\{1, 2\}$ (d) $\{0, 1, 2\}$

22. Let $f(x) = (\sin^{-1}x)^2 + (\cos^{-1}x)^2$ then

(a) Greatest value of $f(x)$ is $\frac{5\pi^2}{8}$

(b) Greatest value of $f(x)$ is $\frac{7\pi^2}{4}$

(c) Least value of $f(x)$ is $\frac{\pi^2}{8}$

(d) Least value of $f(x)$ is $\frac{\pi^2}{12}$

23. The greatest and least values of $(\sin^{-1}x)^3 + (\cos^{-1}x)^3$ are

(a) $\frac{\pi^3}{32},\frac{7\pi^3}{32}$ (b) $\frac{7\pi^3}{8},\frac{\pi^3}{32}$

(c) $\frac{-\pi^3}{8},\frac{7\pi^3}{8}$ (d) $\frac{\pi^3}{8},\frac{\pi^3}{32}$

SECTION-II

More than One Correct Answer Type

24. Let $f(x)$ be a real valued function such that $f(0)=\frac{1}{2}$ and $f(x + y) = f(x) f(a - y) + f(y) f(a - x)\ \forall\ x, y \in R$ then for some real 'a'

(a) $f(x)$ is a periodic function

(b) $f(x)$ is a constant function

(c) $f(x)=\frac{1}{2}$ (d) $f(x)=\frac{\cos x}{2}$

25. If $f(x) = \cos[\pi^2]\, x + \cos[-\pi^2]\, x$, where [.] is greatest integer function, then

(a) $f\left(\frac{\pi}{2}\right) = -1$ (b) $f(\pi) = 1$

(c) $f(-\pi) = 0$ (d) $f\left(\frac{\pi}{4}\right) = 1$

26. Which of the following functions are periodic ?

(a) $f(x) = \text{sgn}\ (e^{-x})$

(b) $f(x) = \begin{cases} 1, \text{ if } x \text{ is a rational number} \\ 0, \text{ if } x \text{ is an irrational number} \end{cases}$

(c) $f(x) = \sqrt{\frac{8}{1+\cos x} + \frac{8}{1-\cos x}}$

(d) $f(x) = \left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] + 2[-x]$ (where $[\cdot]$ denotes greatest integer function).

27. Range of $f(x) = \log_{\sqrt[3]{10}}(\sqrt{5}(2\sin x + \cos x) + 5)$ is

(a) [0, 1] (b) [0, 3]

(c) $\left(-\infty, \frac{1}{3}\right]$ (d) none of these

28. Which of the following functions $f : R \to R$ are bijective ?

(a) $f(x) = x \sin x$ (b) $f(x) = x - \sin^2 x$

(c) $f(x) = x + \sqrt{x^2}$ (d) $f(x) = -x + \cos^2 x$

29. If $f : R \to R$ given by $f(x) = x|x|$, then f is

(a) one-one (b) onto

(c) bijective (d) one-one but not onto

30. The function $f(x) = \frac{|x-1|}{x^2}$ is

(a) one-one in $(2, \infty)$ (b) one-one in $(0, 1)$

(c) one-one in $(-\infty, 0)$ (d) one-one in $(1, 2)$

31. π is the fundamental period of

(a) $\frac{1+\sin x}{\cos x\,(1+\text{cosec}\, x)}$ (b) $|\sin x| + |\cos x|$

(c) $\sin 2x + \cos 2x$ (d) $\cos(\sin x) + \cos(\cos x)$

32. A function $f : R \to R$ is defined by
$f(x + y) - kxy = f(x) + 2y^2\ \forall\ x, y \in R$
and $f(1) = 2; f(2) = 8$, where k is some constant, then

$f(x+y) \cdot f\left(\frac{1}{x+y}\right)$ is equal to (where $x + y \neq 0$)

(a) 1 (b) 4 (c) k (d) $f(1)$

SECTION-III

Comprehension Type

Paragraph for Question No. 33 and 34

Consider the function $f : R \to (0, \infty)$ defined by $f(x) = 2^x + 2^{|x|}$

33. $f(x)$ is

(a) one-one onto (b) one-one into

(c) many-one onto (d) many-one into

34. The number of solutions of the equation $f(x) = \ln \pi$ is

(a) 0 (b) 1 (c) 2 (d) 3

Paragraph for Question No. 35 to 37

Let $f : R - \{0, 1\} \to R$ be a function satisfying the relation $f(x) + f\left(\frac{x-1}{x}\right) = x$ for all $x \in R - \{0, 1\}$. Based on this, answer the following questions.

35. $f(x)$ is equal to

(a) $\frac{1}{2}\left[x + \frac{1}{1-x} - \frac{x-1}{x}\right]$ (b) $\frac{1}{2}\left[x - \frac{1}{1-x} + \frac{x-1}{x}\right]$

(c) $\frac{1}{2}\left[x - \frac{1}{1-x} - \frac{x-1}{x}\right]$ (d) $\frac{1}{2}\left[x + \frac{1}{1-x} + \frac{x-1}{x}\right]$

36. $f(-1)$ is equal to

(a) $\frac{3}{4}$ (b) $\frac{-3}{4}$ (c) $\frac{5}{4}$ (d) $\frac{-5}{4}$

37. $f\left(\frac{1}{2}\right)$ is equal to

(a) $\frac{5}{4}$ (b) $\frac{-7}{4}$ (c) $\frac{7}{4}$ (d) $\frac{9}{4}$

Paragraph for Question No. 38 to 40

$f(x)$ is a polynomial function $f : R \to R$ such that $f(2x) = f'(x) \cdot f''(x)$.

38. The value of $f(3)$ is

(a) 4 (b) 12 (c) 15 (d) 18

39. $f(x)$ is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

40. Equation $f(x) = x$ has

(a) three real and distinct roots

(b) one real root

(c) four real and distinct roots

(d) two real and distinct roots

Paragraph for Question No. 41 and 42

Let $f: N \to N$ (N being the set of positive integers) be a function defined by $f(x)$ = the biggest positive integer obtained by reshuffling the digits of x. For example $f(296) = 962$.

41. f is

(a) one-one and onto (b) one-one and into

(c) many-one and onto (d) many-one and into

42. The biggest positive integer which divides $f(n) - n$, for all $n \in N$, is

(a) 3 (b) 9 (c) 18 (d) 27

SECTION-IV

Matrix-Match Type

43.

Column-I		Column-II	
(A)	Domain of $f(x) = \log[ax^3 + (b+a)x^2 + (b+c)x + c]$ If $b^2 - 4ac < 0$, $a > 0$ is	(p)	$\left[-1, \frac{5}{4}\right]$
(B)	Domain of $f(x) = \ln(\tan^{-1}((x^3 - 6x^2 + 11x - 6)x(e^x - 1)))$ is	(q)	$R - \left\{\frac{1}{5}, 1\right\}$
(C)	Range of $f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$ is	(r)	$(-1, \infty)$
(D)	Range of $f(x) = \sin^2\frac{x}{4} + \cos\frac{x}{4}$ is	(s)	$(1,2) \cup (3,\infty)$

44. Let $f, g : R \to R$ be the function defined by $f(x) = x^2 + 1$ and $g(x) = 2[x] - 1$, where $[x]$ is the largest integer $\le x$. Then match the items given in Column I with those in Column II.

Column-I		Column-II	
(A)	$(gof)\left(\frac{1}{2}\right)$	(p)	3
(B)	$(fog)\left(\frac{3}{2}\right)$	(q)	0
(C)	$(fogof)\left(\frac{3}{4}\right)$	(r)	2
(D)	$(gofog)\left(\frac{2}{3}\right)$	(s)	1

45.

Column-I		Column-II	
(A)	If f is a function such that $f(0) = 2, f(1) = 3$ and $f(x+2) = 2f(x) - f(x+1)$, then $f(5)$ is equal to	(p)	4
(B)	If $f(x) = \begin{cases} x^2, & \text{for } x \ge 0 \\ x, & \text{for } x < 0 \end{cases}$, then $f(\sqrt{13}) =$	(q)	3
(C)	If $f(x) + 2f(1-x) = x^2 + 2$ for all $x \in R$, then $f(5)$ is	(r)	12
(D)	If $f(x) = \frac{4^x}{4^x + 2}$ for all $x \in R$, then $\sum_{k=1}^{6} f\left(\frac{k}{7}\right) =$	(s)	13

46. Match the following

Column-I (Function) Here $[\cdot]$ denote g.i.f, $\{\cdot\}$ denote fractional part		Column-II (fundamental period)	
(A)	$f(x) = e^{\cos^4 \pi x + x - [x] + \cos^2 \pi x}$	(p)	$\frac{1}{3}$
(B)	$f(x) = \cos(2\pi\{2x\}) + \sin(2\pi\{2x\})$	(q)	$\frac{1}{4}$
(C)	$f(x) = \sin(3\pi\{x\}) + \tan(\pi[x])$	(r)	$\frac{1}{2}$
(D)	$f(x) = 3x - [3x + a] - b$, where $a, b \in R^+$	(s)	1

SECTION-V

Integer Answer Type

47. Let $f(x, y)$ be a periodic function satisfying the condition $f(x, y) = f(2x + 2y, 2y - 2x)\ \forall\ x, y \in R$ and let $g(x)$ be a function defined as $g(x) = f(2^x, 0)$. If T is period of $g(x)$, then find the value of $T/4$.

48. If $[\sin x] + \left[\frac{x}{2\pi}\right] + \left[\frac{2x}{5\pi}\right] = \frac{9x}{10\pi}$ then the number of solutions in the interval (30, 40) is (where [.] is G.I.F.)

49. If 'f' is a polynomial function satisfying the condition $f(\tan x) + f(\cot x) = f(\tan x)\cdot f(\cot x)\ \forall x \in \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) - \{0\}$ and $f(2) = 9$, then the value of $\frac{f'(2)}{6}$ is

50. f is a function such that $f(x) + 2f(1 - x) = x^2 + 1$. Value of $f(3)$ is

51. If n is a natural number and $1 \le n \le 100$, then the number of solutions of $\left[\frac{n}{2}\right]+\left[\frac{n}{3}\right]+\left[\frac{n}{5}\right]=\frac{n}{2}+\frac{n}{3}+\frac{n}{5}$ (where [.] denotes the greatest integer function) is

52. If f is a polynomial function satisfying $2 + f(x)\cdot f(y) = f(x) + f(y) + f(xy)\ \forall\ x, y \in R$ and if $f(2) = 5$, then the value of $f(f(1))$ is

53. Let

$$f\left(x+\frac{1}{y}\right)+f\left(x-\frac{1}{y}\right)=2f(x)f\left(\frac{1}{y}\right)\forall x, y \in R,$$

$y \neq 0$ and $f(0) = 0$ then the value of $f(1) + f(2) =$

SOLUTIONS

1. (b) : $2^x + 2^{-x}$ is always > 0 *i.e.*, domain is R

$$y=\frac{2^x-2^{-x}}{2^x+2^{-x}}=\frac{2^{2x}-1}{2^{2x}+1}$$

$$\Rightarrow \frac{1+y}{1-y}=\frac{2\cdot 2^{2x}}{2} \quad \text{(Componendo and Dividendo)}$$

$$= 2^{2x} > 0$$

$$\Rightarrow \frac{1+y}{1-y}>0 \quad i.e., \quad \frac{(1+y)^2}{1-y^2}>0$$

$$\Rightarrow 1 - y^2 > 0 \quad \Rightarrow \quad -1 < y < 1$$

2. (d) : $f(x)=1-\sin^2\frac{x}{4}+\sin\frac{x}{4}$

$$=-\left\{\sin^2\frac{x}{4}-\sin\frac{x}{4}\right\}+1=-\left\{\left(\sin\frac{x}{4}-\frac{1}{2}\right)^2-\frac{1}{4}\right\}+1$$

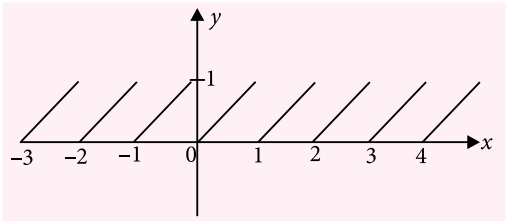
$$=\frac{5}{4}-\left(\sin\frac{x}{4}-\frac{1}{2}\right)^2$$

$$\text{Maximum } f(x)=\frac{5}{4}$$

$$\text{Minimum } f(x)=\frac{5}{4}-\left(-1-\frac{1}{2}\right)^2=\frac{5}{4}-\frac{9}{4}=-1$$

$$\therefore \text{ Range of } f(x)=\left[-1, \frac{5}{4}\right]$$

3. (b) : The graph of $y = x - [x]$ is as shown below.



When x is an integer, $x - [x] = 0$

$[\because\ x \to [x]$ as x tends to an integer.]

Hence, $f(x) = 0$ when x is an integer

As $x \to 1, \frac{x}{1+x} \to \frac{1}{2}$

Hence, the range of $f(x)$ is $\left[0, \frac{1}{2}\right)$.

4. (b) : $[x + 2] \neq 0 \Rightarrow [x] + 2 \neq 0 \Rightarrow [x] \neq -2$

$\Rightarrow$ x should not belong to $[-2, -1)$.

$\therefore$ Domain of f is $(-\infty, -2) \cup [-1, \infty)$.

5. (a) : We have, $f(x)=x^2+1+\frac{1}{x^2+1}-1$

$$\Rightarrow x^2+1+\frac{1}{x^2+1}\ge 2 \quad [\because \text{ A.M.} \ge \text{G.M.}]$$

$$\Rightarrow x^2+\frac{1}{x^2+1}\ge 1 \quad \therefore f(x) \in [1, \infty)$$

6. (b) : Let $y = f(x) = (5 - (x - 8)^5)^{1/3}$, then

$y^3 = 5 - (x - 8)^5 \Rightarrow (x - 8)^5 = 5 - y^3$

$\Rightarrow x = 8 + (5 - y^3)^{1/5}$

Let, $z = g(x) = 8 + (5 - x^3)^{1/5}$

Now, $f(g(x)) = [5 - (x - 8)^5]^{1/3}$

$= (5 - [(5 - x^3)^{1/5}]^5)^{1/3} = (5 - 5 + x^3)^{1/3} = x$

Similarly, we can show that $g(f(x)) = x$

Hence, $g(x) = 8 + (5 - x^3)^{1/5}$ is the inverse of $f(x)$.

7. (a) : $f(x) = |x - 1| + |x - 2| + |x + 1| + |x + 2|$

when $x \in [-2, -1]$

$f(x) = -(x - 1) - (x - 2) - (x + 1) + x + 2 = -2x + 4$

when $x \in [-1, 1]$,

$f(x) = -(x - 1) - (x - 2) + x + 1 + x + 2$

$= -x + 1 - x + 2 + x + 1 + x + 2 = 6$

when $x \in [1, 2]$,

$f(x) = (x - 1) - (x - 2) + x + 1 + x + 2 = 2x + 4$

Plotting the graph of the function, range of $f(x) = [6, 8]$

8. (c) : We have,

$$f(x)=\frac{x^2+x+2}{x^2+x+1}=\frac{(x^2+x+1)+1}{x^2+x+1}=1+\frac{1}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}$$

We can see here that as $x \to \infty$, $f(x) \to 1$ which is the min. value of $f(x)$. Also, $f(x)$ is max. when $\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$ is min. which is so when $x=-\frac{1}{2}$ and then $\frac{3}{4}$.

$\therefore \quad f_{\max}=1+\frac{1}{3/4}=\frac{7}{3} \quad \therefore \quad R_f=\left(1,\frac{7}{3}\right]$

9. (a) : We have $f(x) = -x^3 + x^2 - x + \cos^{-1}x$

Domain = $[-1, 1]$

$$f'(x)=-3x^2+2x-1-\frac{1}{\sqrt{1-x^2}}<0$$

$\therefore$ 'f' is a decreasing function

$\therefore$ Min. of $f(x)$ is $f(1) = -1 + 1 - 1 + 0 = -1$

Max. of $f(x)$ is $f(-1) = 1 + 1 + 1 + \pi = 3 + \pi$

$\therefore$ Range = $[-1, 3 + \pi]$

10. (a) : Let $f(x)=\log_{10}\left\{\sin^{-1}(\sqrt{x-5})+\frac{3\pi}{2}\right\}$

The function is defined if (i) $x - 5 \geq 0$

(ii) $-1\leq\sqrt{x-5}\leq 1$ and (iii) $\sin^{-1}(\sqrt{x-5})+\frac{3\pi}{2}>0$

Now (i) $\Rightarrow x \geq 5$

(ii) $\Rightarrow 0 \leq x - 5 \leq 1 \Rightarrow 5 \leq x \leq 6$

(iii) is satisfied by virtue of (ii).

Hence, considering (i) and (ii), we find that the domain of the function is $D_f = [5, 6]$.

Let $y_1=\sin^{-1}(\sqrt{x-5})$ and $y_2=\sin^{-1}(\sqrt{x-5})+\frac{3\pi}{2}$

so that $y = \log_{10}(y_2)$ where $y_2=y_1+\frac{3\pi}{2}$

Now, for y_1, since $x \in [5, 6]$, $y_1 \geq 0$ so that

$$0\leq y_1\leq\frac{\pi}{2}\left(\because -\frac{\pi}{2}\leq\sin^{-1}(x)\leq\frac{\pi}{2}\right)$$

Consequently, $0+\frac{3\pi}{2}\leq y_1+\frac{3\pi}{2}\leq\frac{\pi}{2}+\frac{3\pi}{2}$

$$\Rightarrow \quad \frac{3\pi}{2}\leq y_2\leq 2\pi$$

$\because$ log is an increasing function in given domain

$$\Rightarrow \quad \log\left(\frac{3\pi}{2}\right)\leq\log(y_2)\leq\log(2\pi)$$

Hence, the range of $f(x)$ is $\left[\log\frac{3\pi}{2},\ \log 2\pi\right]$

11. (d) : $x - 3 \geq 0 \Rightarrow x \geq 3$

$x-2-2\sqrt{x-3}\geq 0$ for $x \geq 3$

$\Rightarrow \quad x-2\geq 2\sqrt{x-3}$ and $x-2+2\sqrt{x-3}\geq 0$

$\Rightarrow \quad x^2 - 8x + 16 \geq 0 \Rightarrow (x - 4)^2 \geq 0 \ \forall \ x \in R$

Hence, Domain = $[3, \infty)$

12. (b) : Let $t=x^3(x^3+3); t=\left(x^3+\frac{3}{2}\right)^2-\frac{9}{4}\in[-\frac{9}{4},\infty)$

$f(x) = g(t) = t(t + 2) = (t + 1)^2 - 1$ is least when $t = -1$

and $-1\in[-\frac{9}{4},\infty)$ $\therefore$ min $f(x)=-1$

13. (a) : The function is defined if (i) sgn $(x^2) > 0$ and (ii) $[x] > 0$ but $[x] \neq 1$.

We know that sgn $(x^2)=\begin{cases}1, \text{if } x^2>0\\ 0, \text{ if } x=0\\ -1, \text{ if } x^2<0\end{cases}$

(i) Since sgn (x^2) is non-negative, we have

$x^2 > 0 \Rightarrow x \in R - \{0\}$ ...(A)

(ii) $\Rightarrow [x] = 2, 3, 4, \ldots \ \therefore \ x \in [2, \infty)$ ...(B)

Hence, $D_f = A \cap B = [2, \infty)$

14. (b) : The function $f(x)$ is defined if

(i) $x^2 - 5x + 10 > 0$

(ii) $1 - \log_{10}(x^2 - 5x + 10) > 0$

Now, (ii) $\Rightarrow \log_{10}(x^2 - 5x + 10) < 1 \Rightarrow x^2 - 5x + 10 < 10$

$\Rightarrow \quad x^2 - 5x < 0 \Rightarrow x(x - 5) < 0 \Rightarrow \ 0 < x < 5$ ...(A)

Again, $x^2 - 5x + 10 > 0$ for all x, ...(B)

Since the discriminant of the corresponding equation $x^2 - 5x + 10 = 0$ is negative, so that the roots of the equation are imaginary.

Combining (A) and (B), we find that the domain of $f(x)$ is $(0, 5)$.

15. (b)

16. (a) : The function is defined if (i) $[x] > 0$ and $[x] \neq 1$ (ii) $\frac{|x|}{x}>0$

(iii) $\log_{[x]}\frac{|x|}{x}\geq 0$ (iv) $0\leq\sqrt{\log_{[x]}\frac{|x|}{x}}\leq 1$

Now, (i) $[x] = 2, 3, \ldots$ *i.e.* $[x] \geq 2 \Rightarrow x \geq 2$ *i.e.* the domain of the function is $[2, \infty)$.

For this value of x (≥ 2) (ii) is true,

(iii) is also true and $\sqrt{\log_{[x]}\frac{|x|}{x}}=\sqrt{\log_{[x]}1}=0$

Hence $f(x)=\cos^{-1}(0)=\frac{\pi}{2}$

Hence, the range of the function is $\left\{\frac{\pi}{2}\right\}$.

17. (c) : Let $f(x)=x^2+x+\frac{3}{4}=\left(x+\frac{1}{2}\right)^2+\frac{1}{2}\geq\frac{1}{2}$

$g(f(x)) = (f(x))^2 + a\,f(x) + 1$, for $g(f(x)) = 0$, we have

$$a=-\left(f(x)+\frac{1}{f(x)}\right)\leq -2$$

$\therefore$ If $a > -2$, $g(f(x)) = 0$ has no solution.

18. (d) : For $0 \le f(x) \le 1$, $g(x) = 0$

For $-1 \le f(x) < 0$

$$g(x) = \frac{e^{2f(x)} - 1}{e^{2f(x)} + 1} \Rightarrow g(x) \in \left[\frac{1-e^2}{1+e^2}, 0\right)$$

$$\therefore \quad \text{Range of } g(x) = \left[\frac{1-e^2}{1+e^2}, 0\right]$$

19. (d) : It is given that $2^x + 2^y = 2 \ \forall \ x, y \in R$

Therefore, $2^x = 2 - 2^y < 2 \Rightarrow 0 < 2^x < 2$

Taking log on both sides with base 2.

$\Rightarrow \log_2 0 < \log_2 2^x < \log_2 2$

Hence, domain is $-\infty < x < 1$

20. (a) : $e^{f(x)} = e - e^x \Rightarrow f(x) = \log_e (e - e^x)$

$e - e^x > 0 \Rightarrow e^1 > e^x \Rightarrow x < 1$

$D_f = (-\infty, 1)$

Let $y = f(x) = \log_e (e - e^x) \Rightarrow e^y = e - e^x$

$\Rightarrow e^x = e - e^y \Rightarrow x = \log_e (e - e^y)$

$\Rightarrow e - e^y > 0 \Rightarrow e^1 > e^y \quad \therefore \quad y < 1$

$R_f = (-\infty, 1)$

21. (d) : We have, $y = [x^2] - [x]^2$, $x \in [0, 2]$

$$i.e., \ y = \begin{cases} [x^2], & 0 \le x < 1 \\ [x^2] - 1, & 1 \le x < 2 \end{cases}$$

$$\text{Now, } y = \begin{cases} [x^2] - 4, & x = 2 \\ 0, & x = 2 \end{cases}$$

$$i.e., \ y = \begin{cases} 0, & 0 \le x < 1 \\ 1 - 1 = 0, & 1 \le x < \sqrt{2} \\ 2 - 1 = 1, & \sqrt{2} \le x < \sqrt{3} \\ 3 - 1 = 2, & \sqrt{3} \le x < 2 \\ 0, & x = 2 \end{cases}$$

Hence, the range is $\{0, 1, 2\}$

22. (c) : $f(x) = 2(\sin^{-1} x)^2 - \pi \sin^{-1} x + \frac{\pi^2}{4}$

$$= 2\left(\sin^{-1} x - \frac{\pi}{4}\right)^2 + \frac{\pi^2}{8} \Rightarrow f(x) \in \left[\frac{\pi^2}{8}, \frac{5\pi^2}{4}\right]$$

23. (b) :

$$(\sin^{-1} x)^3 + (\cos^{-1} x)^3 = \left(\frac{\pi}{2}\right)^3 - 3\sin^{-1} x \cos^{-1} x \left(\frac{\pi}{2}\right)$$

$$= \frac{\pi^3}{8} - \frac{3\pi}{2} \sin^{-1} x \left(\frac{\pi}{2} - \sin^{-1} x\right)$$

$$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left\{\sin^{-1} x - \frac{\pi}{4}\right\}^2$$

$$\text{minimum} = \frac{\pi^3}{32}, \text{maximum} = \frac{7\pi^3}{8}$$

24. (a, b, c) : $f(x + y) = f(x) f(a - y) + f(y) f(a - x)$ ...(1)

Put $x = y = 0$, we get $f(a) = \frac{1}{2}$

Put $y = 0$, we get

$f(x) = f(x) \cdot f(a) + f(0) f(a - x) \Rightarrow f(x) = f(a - x)$

Put $y = a - x$ in equation (1), we get

$f(a) = (f(x))^2 + (f(a - x))^2$

$$\Rightarrow \quad f(x) = \pm\frac{1}{2}\left\{f(x) \ne -\frac{1}{2}\right\}$$

Hence, $f(x) = \frac{1}{2}$

25. (a, c) : $9 < \pi^2 < 10 \Rightarrow [\pi^2] = 9$

$-10 < -\pi^2 < -9 \Rightarrow [-\pi^2] = -10$

$f(x) = \cos 9x + \cos(-10)x = \cos 9x + \cos 10x$

$$f\left(\frac{\pi}{2}\right) = \cos 5\pi = -1, \ f(\pi) = 0, \ f(-\pi) = 0$$

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

26. (a, b, c, d) : (a) $f(x) = \text{sgn}\ (e^{-x})$

$e^{-x} > 0$ for every real x

Every constant function is a periodic function.

(b) Let $T > 0$ be a rational number then

$$f(x + T) = \begin{cases} 1, & x \in Q \\ 0, & x \notin Q \end{cases} = f(x)$$

$f(x)$ is a periodic function.

(c) $$f(x) = \sqrt{\frac{8}{1+\cos x} + \frac{8}{1-\cos x}}$$

$$= \sqrt{\frac{16}{1-\cos^2 x}} = \frac{4}{|\sin x|} = 4|\text{cosec } x|$$

This is a periodic function.

(d) $$f(x) = \left[x + \frac{1}{2}\right] + \left[x - \frac{1}{2}\right] + 2[-x]$$

$$= \left(x + \frac{1}{2} - \left\{x + \frac{1}{2}\right\} + x - \frac{1}{2} - \left\{x - \frac{1}{2}\right\}\right)$$

$$+ 2\left(-x - \{-x\}\right) = -\left\{x + \frac{1}{2}\right\} - \left\{x - \frac{1}{2}\right\} - 2\{-x\}$$

Since $\{.\}$ is a periodic function, hence this function is periodic.

27. (d) : We know that

$-\sqrt{5} \le 2\sin x + \cos x \le \sqrt{5}, \ \forall x \in R$

$\Rightarrow \quad -5 \le \sqrt{5}(2\sin x + \cos x) \le 5$

$\Rightarrow \quad 0 \le \sqrt{5}(2\sin x + \cos x) + 5 \le 10$

$\Rightarrow \quad -\infty < \log_{\sqrt{10}}(\sqrt{5}(2\sin x + \cos x) + 5) \le 3$

Hence range is $(-\infty, 3)$

28. (b, d) : (a) f is not one-one as f cuts x-axis twice in $(0, 2\pi]$

In fact f is continuous and achieves every real number infinite times.

(b) f is monotonic as $f'(x) = 1 - \sin 2x \geq 0 \ \forall \ x \in R$

and $\lim\limits_{x \to -\infty} f = -\infty$; $\lim\limits_{x \to \infty} f = \infty$ $\therefore$ f is continuous.

Hence, f is bijective.

(c) $f(x) = x + |x| = \begin{cases} 2x, & \text{if} \quad x \geq 0 \\ 0, & \text{if} \quad x \leq 0 \end{cases}$ is not one-one as f is a constant function for $x \leq 0$

(d) f is monotonic as $f'(x) \leq 0 \ \forall \ x \in R$ and as in (b), f is bijective.

29. (a, b, c) : $f(x) = \begin{cases} x^2, & x \geq 0 \\ 0, & x < 0 \end{cases}$

$x^2 \geq 0, -x^2 < 0 \implies$ Range $= R \implies f$ is onto.

For $x > 0, x_1^2 = x_2^2 \implies x_1 = x_2$

For $x < 0, -x_1^2 = -x_2^2 \implies x_1 = x_2$, Hence f is one-one.

30. (a, b, c, d) : $f(x) = \dfrac{|x-1|}{x^2} = \begin{cases} \dfrac{1-x}{x^2}, & x < 1, x \neq 0 \\ \dfrac{x-1}{x^2}, & x \geq 1 \end{cases}$

$$\implies f'(x) = \begin{cases} \dfrac{x-2}{x^3}, & x < 1, x \neq 0 \\ \dfrac{2-x}{x^3}, & x > 1 \end{cases}$$

$$f'(x) < 0 \implies \left.\begin{array}{l} \dfrac{x-2}{x^3} < 0, x < 1, x \neq 0 \\ \dfrac{2-x}{x^3} < 0, x > 1 \end{array}\right\}$$

$\implies x \in (-\infty, 0) \cup (0, 1) \cup (1, 2) \cup (2, \infty)$

31. (a, c) : Let $f(x) = \dfrac{1+\sin x}{\cos x(1+\operatorname{cosec} x)}$

$$= \frac{(1+\sin x)\sin x}{\cos x(1+\sin x)} = \tan x, x \neq \frac{n\pi}{2}$$

Clearly, $f(x)$ has π as fundamental period.

Let $f(x) = |\sin x| + |\cos x|$

$f(\pi + x) = |\sin x| + |\cos x| = f(x)$

$$f\left(\frac{\pi}{2} + x\right) = |\cos x| + |\sin x| = f(x)$$

$\implies \dfrac{\pi}{2}$ is the fundamental period.

Let $f(x) = \sin 2x + \cos 2x$

$f(\pi + x) = \sin 2x + \cos 2x = f(x)$

$$f\left(\frac{\pi}{2} + x\right) = -\sin 2x - \cos 2x \neq f(x)$$

$\implies \pi$ is the fundamental period.

Let $f(x) = \cos(\sin x) + \cos(\cos x)$

$f(\pi + x) = \cos(\sin x) + \cos(\cos x)$

$$f\left(\frac{\pi}{2} + x\right) = \cos(\cos x) + \cos(\sin x)$$

$\implies$ fundamental period is $\dfrac{\pi}{2}$.

32. (b, c) : Given, $f(x + y) - kxy = f(x) + 2y^2$

Replace y by $-x$, then $f(0) + kx^2 = f(x) + 2x^2$

$\implies f(x) = f(0) + kx^2 - 2x^2$ ...(1)

Now, $f(1) = f(0) + k - 2 = 2 \implies f(0) = -k + 4$

and $f(2) = f(0) + 4k - 8 = 8 \implies f(0) = -4k + 16$

Which give $k = 4$ and $f(0) = 0$

Thus, from (1), $f(x) = 2x^2$

$$\therefore \quad f(x+y) f\left(\frac{1}{x+y}\right) = 4 = k$$

33. (d)

34. (a) : We have $f(x) = 2^x + 2^{|x|}$

For $x \geq 0, f(x) = 2 \cdot 2^x$

And for $x < 0; f(x) = 2^x + 2^{-x}$

and in this case

$$f'(x) = 2^x \ln 2 - 2^{-x} \ln 2 = \ln 2 \left(\frac{2^{2x} - 1}{2^x}\right)$$

$$= \frac{\ln 2}{2^x}.(2^x + 1)(2^x - 1) < 0$$

So, $f(x)$ is decreasing in $x \in (-\infty, 0)$ and increasing in $x \in (0, \infty)$.

So, $f(x)$ is many – one.

Also, Range of $f(x)$ is $[2, \infty)$

$\therefore$ f is many – one into.

Since $\ln \pi < 2$

So, the equation $2^x + 2^{|x|} = \ln \pi$ has no solution.

35. (a) **36. (d)** **37. (c)**

(35-37) : Given that, $f(x) + f\left(\dfrac{x-1}{x}\right) = x$ ....(1)

For all $x \neq 0, 1$ replacing x with $\dfrac{x-1}{x}$ on both sides, we get

$$f\left(\frac{x-1}{x}\right) + f\left(\frac{[(x-1)/x] - 1}{(x-1)/x}\right) = \frac{x-1}{x}$$

That is $f\left(\dfrac{x-1}{x}\right) + f\left(\dfrac{1}{1-x}\right) = \dfrac{x-1}{x}$ .....(2)

Again replacing x with $\dfrac{x-1}{x}$, we get

$$f\left(\frac{1}{1-x}\right) + f(x) = \frac{1}{1-x}$$ .....(3)

Then by taking eq. (1) + eq. (3) – eq. (2), we get

$$2f(x)=x+\frac{1}{1-x}-\frac{x-1}{x}$$

or $f(x)=\frac{1}{2}\left[x+\frac{1}{1-x}-\frac{x-1}{x}\right]$ .....(4)

Substituting $x=-1$ and $x=\frac{1}{2}$ in eq. (4) we get the solutions for Q.No. 36 and Q. No. 37.

38. (b) : Suppose degree of $f(x)$ is n, then the degree of f' is $(n-1)$ and degree of f'' is $(n-2)$.

So, $n=(n-1)+(n-2) \Rightarrow n=3$

Hence, $f(x)=ax^3+bx^2+cx+d$

Given, $f(2x)=f'(x)\cdot f''(x)$

$\Rightarrow 8ax^3+4bx^2+2cx+d=(3ax^2+2bx+c)(6ax+2b)$

Comparing coefficients of terms, we have

$a=4/9, b=0, c=0$ and $d=0$

$\Rightarrow f(x)=\frac{4x^3}{9} \Rightarrow f(3)=12$

39. (a) one-one and onto

40. (a) $\frac{4x^3}{9}=x \Rightarrow x=0,\pm\frac{3}{2}$

41. (d) : As $f(296)=f(962)$, f is many-one.

Further the numbers whose digits increase from left to right (for example) have no pre-image.

Hence f is into.

42. (b) : It is easy to see that the remainder, when a positive integer is divided by 9, is the same as the sum of the digits of the number (until the sum becomes a one digit number). Thus $f(n)$ and n leave the same remainder, when divided by 9. Hence 9 divides $f(n)-n$. Further there is no reason to expect that the number is divisible by 27. The number $f(n)-n$ is not divisible by 18 also, in case $f(n)-n$ is odd.

Hence 9 is the biggest number.

By the definition of f, digits of $f(x)$ are non-increasing from left to right.

43. A → r, s; B → s; C → q, s; D → p

(A) $f(x)=\log(x(ax^2+bx+c)+(ax^2+bx+c)$.

$=\log(x+1)(ax^2+bx+c)$

$\therefore$ domain of $f(x)=(-1,\infty)$

as $ax^2+bx+c>0 \ \forall x$ (since $a>0, \Delta<0$)

(B) For $\ln(\tan^{-1}(x-1)(x-2)(x-3)x(e^x-1))$ to be defined $x(e^x-1)(x-1)(x-2)(x-3)>0$

$\Rightarrow x\in(1,2)\cup(3,\infty)$

(C) $y=f(x)=\frac{x-1}{x+3}$ for $x\neq 2,-3$

and $x=\frac{3y+1}{1-y}$ $y\neq 1$ and $x\to 2\Rightarrow y\to\frac{1}{5}$

$\therefore$ Range of $f(x)=R-\left\{1,\frac{1}{5}\right\}$

(D) $f(x)=1-\cos^2\frac{x}{4}+\cos\frac{x}{4}=\frac{5}{4}-\left(\cos\frac{x}{4}-\frac{1}{2}\right)^2$

$\Rightarrow -1\le f(x)\le\frac{5}{4}$

44. A → s; B → r; C → r; D → p

(A) $(gof)\left(\frac{1}{2}\right)=g\left[f\left(\frac{1}{2}\right)\right]=g\left(\frac{5}{4}\right)=2(1)-1=1$

(B) $(fog)\left(\frac{3}{2}\right)=f\left[g\left(\frac{3}{2}\right)\right]=f(1)=1^2+1=2$

(C) $(fogof)\left(\frac{3}{4}\right)=(fog)\left(f\left(\frac{3}{4}\right)\right)=(fog)\left(\frac{25}{16}\right)=f\left[g\left(\frac{25}{16}\right)\right]$

$=f(1)=1^2+1=2$

(D) $(gofog)\left(\frac{2}{3}\right)=(gof)\left[g\left(\frac{2}{3}\right)\right]=(gof)(-1)=g[f(-1)]$

$=g(2)=2\times 2-1=3$

45. A → s; B → s; C → q; D → q

(A) Given, $f(0)=2, f(1)=3$

and $f(x+2)=2f(x)-f(x+1)$

Put $x=0$

$f(2)=2f(0)-f(1)=(2\times 2)-3=1$

Put $x=1$

$f(3)=2f(1)-f(2)=(2\times 3)-1=5$

Put $x=2$

$f(4)=2f(2)-f(3)=(2\times 1)-5=-3$

Put $x=3$

$f(5)=2f(3)-f(4)=(2\times 5)-(-3)=13$

(B) $\sqrt{13}>0\Rightarrow f(\sqrt{13})=(\sqrt{13})^2=13$

(C) Given, $f(x)+2f(1-x)=x^2+2, \forall x\in R$ ...(1)

Replace x by $1-x$

$f(1-x)+2f(x)=(1-x)^2+2$ ...(2)

Multiply eq (2) by 2

$2f(1-x)+4f(x)=2(1-x)^2+4$ ...(3)

(3) – (1) $\Rightarrow 3f(x)=2(1-x)^2-x^2+2$

Put $x=5$, we get $f(5)=3$

(D) We have $f(x)=\frac{4^x}{4^x+2}$

$$\Rightarrow f(1-x)=\frac{4^{1-x}}{4^{1-x}+2}=\frac{\frac{4}{4^x}}{\frac{4}{4^x}+2}=\frac{4}{4+2\cdot 4^x}=\frac{2}{4^x+2}$$

$\therefore f(x)+f(1-x)=1$

$$\Rightarrow \quad f\left(\frac{1}{7}\right)+f\left(\frac{6}{7}\right)=1 \qquad \text{...(i)}$$

$$f\left(\frac{2}{7}\right)+f\left(\frac{5}{7}\right)=1 \qquad \text{...(ii)}$$

$$f\left(\frac{3}{7}\right)+f\left(\frac{4}{7}\right)=1 \qquad \text{...(iii)}$$

Adding (i), (ii) and (iii), we get

$$\sum_{k=1}^{6} f\left(\frac{k}{7}\right)=1+1+1=3$$

46. **A → s; B → r; C → s; D → p**

(A) $e^{\cos^4 \pi x}+x-[x]+\cos^2 \pi x$

$\cos^4 \pi x$, $\cos^2 \pi x$, $x-[x]$ are periodic with period 1

$\therefore$ Period of $f(x)$ is 1.

(B) $f(x)=\cos 2\pi\{2x\}+\sin 2\pi\{2x\}$

Since $\{2x\}$ is periodic with period 1/2

$\therefore$ Period of $f(x)$ is $\frac{1}{2}$.

(C) $f(x)=\sin 3\pi\{x\}+\tan \pi[x]$

Clearly $\tan \pi[x]=0 \ \forall\ x \in R$ and period of $\{x\}$ is 1

$\sin 3\pi\{x\}$ is periodic with period 1.

(D) For $f(x)=x-[x+a]-b=(x+a)-[x+a]-(a+b)$

$=\{x+a\}-(a+b)$, which is periodic with period 1.

$\therefore$ For $f(x)=3x-[3x+a]-b$

$=(3x+a)-[3x+a]-(a+b)=\{3x+a\}-(a+b)$

Which is periodic with period 1/3

47. (3) : $f(x, y)=f(2x+2y, 2y-2x)=f(2(2x+2y)$

$+2(2y-2x), 2(2y-2x)-2(2x+2y))$

$=f(8y, -8x)$

$=f(8(-8x), -8(8y))=f(-64x, -64y)$

$=f(-64(-64x), -64(-64y))$

$=f(2^{12}x, 2^{12}y)$

$f(x, y)=f(2^{12}x, 2^{12}y)$

$f(x, 0)=f(2^{12}x, 0)$

$g(x)=f(2^x, 0)=f(2^{12}2^x, 0)$

$=f(2^{x+12}, 0)=g(x+12)$

$\Rightarrow \quad g(x)=g(x+12) \Rightarrow$ Period of $g(x)$ is $T=12 \Rightarrow \frac{T}{4}=3$

48. (1) : $[\sin x]=\frac{x}{2\pi}-\left[\frac{x}{2\pi}\right]+\frac{2x}{5\pi}-\left[\frac{2x}{5\pi}\right]$

Number of solution = 1.

49. (2) : Putting $\tan x = t$, we get $f(t)$ is a polynomial function of the form, $f(t) = \pm t^n + 1$

When $t = 2, f(2) = 9 \ \Rightarrow \ n = 3$

$\therefore \ f(t) = t^3 + 1 \ \Rightarrow \ f'(t) = 3t^2$

$f'(2)=12 \quad \therefore \quad \frac{f'(2)}{6}=2$

50. (0) : $f(x) + 2f(1 - x) = x^2 + 1$

$\therefore \ f(1 - x) + 2f(x) = (1 - x)^2 + 1 = x^2 - 2x + 2$

Solving we get, $3f(x) = x^2 - 4x + 3$

$f(3) = 0$

51. (3) : From the given equation, we have

$$\left(\frac{n}{2}-\left[\frac{n}{2}\right]\right)+\left(\frac{n}{3}-\left[\frac{n}{3}\right]\right)+\left(\frac{n}{5}-\left[\frac{n}{5}\right]\right)=0$$

$$\Rightarrow \left\{\frac{n}{2}\right\}+\left\{\frac{n}{3}\right\}+\left\{\frac{n}{5}\right\}=0;$$ where $\{.\}$ is a fractional part

But each of the fraction part function is positive and their sum is zero. Hence, each of the fraction part function is zero. Consequently, each of $\frac{n}{2}, \frac{n}{3}, \frac{n}{5}$ is an integer. The 1.c.m. of 2, 3, 5 is 30. Therefore we can take $n = 30k$, where k is an integer.

Hence the number of solutions such that $1 \le n \le 100$ is = 3 (n = 30, 60 and 90).

52. (5) : $2 + f(x)\cdot f(y) = f(x) + f(y) + f(xy) \ \forall\ x, y \in R$....(i)

$f(2) = 5$

$$\text{Put } y=\frac{1}{x} \Rightarrow 2+f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)+f(1) \quad \text{...(ii)}$$

Put $x = 1 \ \Rightarrow \ 2 + f(1)\cdot f(1) = f(1) + f(1) + f(1)$

$\Rightarrow \ [f(1)]^2 - 3\cdot f(1) + 2 = 0$

$\Rightarrow \ [f(1) - 2][f(1) - 1] = 0$

As $f(2) = 5 \ \Rightarrow \ f(1) \ne 1 \ \Rightarrow \ f(1) = 2$

$$\text{(ii)} \ \Rightarrow \ 2+f(x)\cdot f\left(\frac{1}{x}\right)=f(x)+f\left(\frac{1}{x}\right)+2$$

$\Rightarrow \ f(x) = \pm x^n + 1$

Since $f(2) = 5 \ \Rightarrow \ n = 2 \ \Rightarrow \ f(x) = x^2 + 1$

$\therefore \ f(f(1)) = f(2) = 5$

53. (0) : Let $f\left(x+\frac{1}{y}\right)+f\left(x-\frac{1}{y}\right)$ ...(i)

$$=2f(x)\cdot f\left(\frac{1}{y}\right) \forall x, y \in R$$

Given $f(0) = 0$

(i) put $x = 0, \ y=\frac{1}{x}$, we get

$f(x) + f(-x) = 2f(0)\, f(x)$

$\Rightarrow \ f(x) + f(-x) = 0 \ \Rightarrow \ f(x) = -f(-x)$ ...(ii)

put $x = 1, y = 1$ in (i) $\Rightarrow f(2) + f(0) = 2[f(1)]^2$

$\Rightarrow \ f(2) = 2[f(1)]^2$ ...(A)

put $x = -1, y = -1$ in (i)

$\Rightarrow \ f(-2) = 2f(-1)f(-1) \ \Rightarrow \ -f(2) = 2(f(1)^2$ ...(B)

For (A) and (B), $f(2) = -f(2) \ \Rightarrow \ f(2) = 0 \ \Rightarrow f(1) = 0$

$\therefore \ f(1) = f(2) = 0 \quad \therefore \ f(1) + f(2) = 0$

COMEDK

SOLVED PAPER 2015

1. If $\tan(x + y) = 33$ and $x = \tan^{-1} 3$, then y is

(a) $\frac{3}{10}$ (b) $\frac{33}{10}$ (c) $\frac{1}{3}$ (d) $\tan^{-1}\frac{3}{10}$

2. $\lim\limits_{x\to\infty}\left(\sqrt{a^2x^2+bx+x}-ax\right)=$

(a) $\frac{b}{2a}$ (b) $\frac{b}{a}$ (c) 0 (d) $\frac{2b}{a}$

3. The solution of $\frac{6x}{4x-1} < \frac{1}{2}$ is

(a) $x < \frac{-1}{8}$ (b) $\frac{-1}{8} < x < \frac{1}{4}$

(c) $x < \frac{-1}{8}$ and $x > \frac{1}{4}$ (d) $x > \frac{1}{8}$

4. If $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $A^T + A = I_2$, then

(a) $\theta = n\pi, n \in Z$ (b) $\theta = (2n+1)\frac{\pi}{2}, n \in Z$

(c) $\theta = 2n\pi \pm \frac{\pi}{3}, n \in Z$ (d) None of these

5. If A and B are 2 matrices such that $AB = A$ and $BA = B$, then B^2 is equal to

(a) B (b) A
(c) zero matrix (d) 1

6. Given 5 line segments of lengths 2, 3, 4, 5, 6 units. Then, the no. of triangles that can be formed by joining these segments is

(a) ${}^5C_3 - 3$ (b) ${}^5C_3$
(c) ${}^5C_3 - 1$ (d) ${}^5C_3 - 2$

7. How many numbers greater than 10,00,000 be formed from 2, 3, 0, 3, 4, 2, 3?

(a) 420 (b) 360
(c) 400 (d) 300

8. An urn contains 9 balls, 2 of which are white, 3 blue and 4 black. 3 balls are drawn at random from the urn. The chance that 2 balls will be of the same colour and the third of a different colour is

(a) $\frac{45}{84}$ (b) $\frac{55}{84}$ (c) $\frac{35}{84}$ (d) $\frac{25}{84}$

9. 6 boys & 6 girls sit in a row at random. The probability that all the girls sit together is

(a) $\frac{1}{432}$ (b) $\frac{12}{431}$

(c) $\frac{1}{132}$ (d) None of these

10. Three dice are rolled once. The chance of getting a score of 5 is

(a) $\frac{5}{216}$ (b) $\frac{1}{6}$ (c) $\frac{1}{36}$ (d) $\frac{1}{7^2}$

11. A bag contains 3 white, 4 black, 2 red balls. If 2 balls are drawn at random, then the probability that both the balls are white, is

(a) $\frac{1}{18}$ (b) $\frac{1}{36}$ (c) $\frac{1}{12}$ (d) $\frac{1}{24}$

12. If two sets A and B have 99 elements in common, then the number of elements common to the sets $A \times B$ and $B \times A$ is

(a) 2^{99} (b) 99^2 (c) 100 (d) 18

13. If $f(x) = e^x$ and $g(x) = \log e^x$, then which of the following is TRUE?

(a) $f\{g(x)\} \neq g\{f(x)\}$ (b) $f\{g(x)\} = g\{f(x)\}$
(c) $f\{g(x)\} + g\{f(x)\} = 0$ (d) $f\{g(x)\} - g\{f(x)\} = 1$

14. The number of bijective functions from the set A to itself, if A contains 108 elements is

(a) 108 (b) (108)! (c) $(108)^2$ (d) 2^{108}

15. If $y = |\cos x| + |\sin x|$, then $\frac{dy}{dx}$ at $x = \frac{2\pi}{3}$ is

(a) $\frac{1-\sqrt{3}}{2}$ (b) 0

(c) $\frac{1}{2}(\sqrt{3}-1)$ (d) None of these

16. If a, b, c are in A.P., then the value of the determinant $\begin{vmatrix} x+2 & x+3 & x+2a \\ x+3 & x+4 & x+2b \\ x+4 & x+5 & x+2c \end{vmatrix}$ is

(a) 0 (b) 1 (c) x (d) $2x$

17. If $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$, then det(adj A) is

(a) a^{27} (b) a^9 (c) a^6 (d) a^2

18. The differential equation of the family of parabolas $y^2 = 4ax$, where a is parameter, is

(a) $\frac{dy}{dx} = \frac{y}{2x}$ (b) $\frac{dy}{dx} = -\frac{y}{2x}$

(c) $\frac{dy}{dx} = -\frac{2y}{x}$ (d) $\frac{dy}{dx} = \frac{2y}{x}$

19. If $\frac{dy}{dx} = \frac{y + x\tan\frac{y}{x}}{x}$, then $\sin\left(\frac{y}{x}\right) =$

(a) cx^2 (b) cx (c) cx^3 (d) $\log x$

20. The general solution of the D.E. $\left(\frac{dy}{dx}\right) + y \cdot g'(x) = g(x) \cdot g'(x)$, where $g(x)$ is a given function of x is

(a) $g(x) + \log(1 + y + g(x)) = C$
(b) $g(x) + \log(1 + y - g(x)) = C$
(c) $g(x) - \log(1 + y - g(x)) = C$
(d) $g(x) - \log(1 - y + g(x)) = C$

21. The product of the degree and order of the D.E. $\left(\frac{d^2y}{dx^2}\right)^2 - \left(\frac{dy}{dx}\right)^3 = y^3$ is

(a) 4 (b) 6 (c) 2 (d) 3

22. $\int_{-1}^{1} (x^{27}\cos x + e^x)dx =$

(a) $\frac{2e-1}{e}$ (b) $\frac{e+1}{e}$ (c) $e - \frac{1}{e}$ (d) $\frac{1}{e}$

23. The value of the integral $\int_0^{\pi} \frac{x\sin^{2n}x}{\sin^{2n}x + \cos^{2n}x}dx$ is

(a) π^2 (b) $2\pi^2$ (c) $\frac{\pi^2}{4}$ (d) $\frac{\pi^2}{2}$

24. $\int \frac{x^3 - 1}{x^3 + x}dx =$

(a) $x - \log x + \log(x^2 + 1) - \tan^{-1}x + c$
(b) $x - \log x + \frac{1}{2}\log(x^2 + 1) - \tan^{-1}x + c$
(c) $x + \log x + \log(x^2 + 1) - \tan^{-1}x + c$
(d) $x + \log x + \frac{1}{2}\log(x^2 + 1) - \tan^{-1}x + c$

25. If $\int \frac{\cos 8x + 1}{\tan 2x - \cot 2x}dx = a\cos 8x + c$, then $a =$

(a) $-\frac{1}{16}$ (b) $\frac{1}{8}$ (c) $\frac{1}{16}$ (d) $-\frac{1}{8}$

26. If $\int_0^{\pi} x\, f(\sin x)\, dx = A\int_0^{\pi/2} f(\sin x)\, dx$, then A is

(a) 0 (b) 2π (c) $\frac{\pi}{4}$ (d) π

27. $\cot^{-1}(21) + \cot^{-1}(13) + \cot^{-1}(-8) =$

(a) 0 (b) $\cot^{-1}126$
(c) π (d) None of these

28. $\tan\left(\cos^{-1}\left(\frac{1}{5\sqrt{2}}\right) - \sin^{-1}\left(\frac{4}{\sqrt{17}}\right)\right)$ is

(a) $\frac{\sqrt{29}}{3}$ (b) $\frac{29}{3}$ (c) $\frac{\sqrt{3}}{29}$ (d) $\frac{3}{29}$

29. $11^3 + 12^3 + 13^3 + \ldots\ldots\ldots\ldots + 20^3$ is

(a) an even integer
(b) an odd integer divisible by 5
(c) multiple of 10
(d) an odd integer but not a multiple of 5

30. If a class of 175 students the following data shows the number of students opting one or more subjects. Mathematics 100, Physics 70, Chemistry 40, Mathematics and Physics 30, Mathematics and Chemistry 28, Physics and Chemistry 23, Mathematics, Physics and Chemistry 18. The number of students who have opted Mathematics alone is

(a) 35 (b) 48 (c) 60 (d) 22

31. If a set A has 4 elements, then the total number of proper subsets of set A, is

(a) 16 (b) 14 (c) 15 (d) 17

32. The mean and variance for the data 6, 7, 10, 12, 13, 4, 8, 12 respectively are

(a) 8, $\sqrt{26.25}$ (b) 9, $\sqrt{9.25}$
(c) 8, 26.25 (d) 9, 9.25

33. The medians AD and BE of a triangle with vertices $A(0, b)$, $B(0, 0)$ & $C(a, 0)$ are perpendicular to each other, if

(a) $a = \frac{b}{2}$ (b) $b = \frac{a}{2}$ (c) $ab = 1$ (d) $a = \pm\sqrt{2}\, b$

34. Let a and b be non-zero reals such that $a \neq b$. Then the equation of the line passing through the origin and the point of intersection of $\frac{x}{a} + \frac{y}{b} = 1$ and $\frac{x}{b} + \frac{y}{a} = 1$ is

(a) $ax + by = 0$ (b) $bx + ay = 0$
(c) $y - x = 0$ (d) $x + y = 0$

35. A straight line meets the coordinates axes at A and B, so that the centroid of the triangle OAB is $(1, 2)$. Then the equation of the line AB is
(a) $x + y = 6$ (b) $2x + y = 6$
(c) $x + 2y = 6$ (d) $3x + y = 6$

36. The direction ratios of the line which is perpendicular to the lines $\frac{x-7}{2} = \frac{y+17}{-3} = \frac{z-6}{1}$ and $\frac{x+5}{1} = \frac{y+3}{2} = \frac{z-4}{-2}$ are
(a) $< 4, 5, 7 >$ (b) $< 4, -5, 7 >$
(c) $< 4, -5, -7 >$ (d) $< -4, 5, 7 >$

37. A line making angles 45° and 60° with the positive directions of the axis of x and y, makes with the positive direction of z-axis, an angle of
(a) 60° (b) 120°
(c) 60° and 120° (d) None of these

38. The number of solutions of the equation $|\cot x| = \cot x + \frac{1}{\sin x}$, $(0 \le x \le 2\pi)$ is
(a) 0 (b) 1 (c) 2 (d) 3

39. $\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x}$ is equal to
(a) $-2 \sin x$ (b) $\frac{2}{\sin x}$ (c) $\frac{1}{\sin x}$ (d) $2 \sin x$

40. In a ΔABC, cosec $A(\sin B \cos C + \cos B \sin C) =$
(a) 0 (b) 1
(c) –1 (d) None of these

41. Given two vectors $\hat{i} - \hat{j}$ and $\hat{i} - 2\hat{j}$. The unit vector, coplanar with the two given vectors and perpendicular to $(\hat{i} - \hat{j})$ is
(a) $\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ (b) $\frac{1}{\sqrt{5}}(2\hat{i} + \hat{j})$
(c) $\pm\frac{1}{\sqrt{2}}(\hat{i} + \hat{j})$ (d) None of these

42. If $\vec{a}, \vec{b}, \vec{c}$ are three non-zero vectors such that each one of them are perpendicular to the sum of the other two vectors, then the value of $|\vec{a} + \vec{b} + \vec{c}|^2$ is
(a) $|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$ (b) $|\vec{a}| + |\vec{b}| + |\vec{c}|$
(c) $2(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2)$ (d) $\frac{1}{2}(|\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2)$

43. The slant height of a cone is fixed at 7 cm. If the rate of increase of its height is 0.3 cm/sec, then the rate of increase of its volume when its height is 4 cm is
(a) $\frac{\pi}{2}$ cc/sec (b) π cc/sec
(c) $\frac{\pi}{5}$ cc/sec (d) $\frac{\pi}{10}$ cc/sec

44. If $S^2 = at^2 + 2bt + c$, then acceleration is
(a) Directly proportional to S
(b) Inversely proportional to S
(c) Directly proportional to S^2
(d) Inversely proportional to S^3

45. A ladder 5 m long is leaning against a wall. The bottom of the ladder is pulled along the ground away from the wall, at the rate of 2m/sec. The speed at which its height on the wall decreases when the foot of the ladder is 4 m away from the wall is
(a) $\frac{3}{8}$ m/sec (b) $\frac{8}{3}$ m/sec
(c) $\frac{5}{3}$ m/sec (d) $\frac{2}{3}$ m/sec

46. The angle between the curves $y^2 = 4ax$ and $ay = 2x^2$ is
(a) $\tan^{-1}\frac{3}{4}$ (b) $\tan^{-1}\frac{3}{5}$
(c) $\tan^{-1}\frac{4}{3}$ (d) $\tan^{-1}\frac{5}{3}$

47. Area bounded by the curve $y = x^3$, the x-axis and the ordinates at $x = -2$ and $x = 1$, is
(a) – 9 sq. units (b) $-\frac{15}{4}$ sq. units
(c) $\frac{15}{4}$ sq. units (d) $\frac{17}{4}$ sq. units

48. The area in square units bounded by the normal at (1, 2) to the parabola $y^2 = 4x$, x-axis and the curve is given by
(a) $\frac{10}{3}$ (b) $\frac{7}{3}$
(c) $\frac{4}{3}$ (d) None of these

49. Area bounded by the y axis, $y = \cos x$, $y = \sin x$ when $0 \le x \le \frac{\pi}{2}$ is
(a) $\sqrt{2} - 1$ (b) $2(\sqrt{2} - 1)$
(c) $(\sqrt{2} + 1)$ (d) $\sqrt{2}$

50. The area in square units of the region bounded by $y^2 = 9x$ and $y = 3x$ is
(a) 2 (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

51. The expression

$$\frac{1}{\sqrt{(3x+1)}}\left[\left(\frac{1+\sqrt{3x+1}}{2}\right)^7-\left(\frac{1-\sqrt{3x+1}}{2}\right)^7\right]$$

is a polynomial in x of degree

(a) 7 (b) 5 (c) 4 (d) 3

52. Amplitude of the complex number $i\sin\left(\frac{\pi}{19}\right)$ is

(a) $\frac{\pi}{19}$ (b) $-\frac{\pi}{19}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{2}-\frac{\pi}{19}$

53. The maximum area in square units of an isosceles triangle inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with its vertex at one end of the major axis is

(a) $\sqrt{3}\,ab$ (b) $\frac{3\sqrt{3}}{4}ab$

(c) $\frac{5\sqrt{3}}{4}ab$ (d) None of these

54. If $f(x+y)=f(x)\,f(y)$ for all x and y and $f(5)=2$, $f'(0)=3$, then $f'(5)$ is

(a) 5 (b) 6
(c) 0 (d) None of these

55. If $f(x)=\log_{x^2}(\log x)$, then $f'(x)$ at $x=e$ is

(a) 0 (b) 1 (c) $\frac{1}{e}$ (d) $\frac{1}{2e}$

56. The value of c in Lagrange's theorem for the function $f(x)=\log(\sin x)$ in the interval $\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{2}$

(c) $\frac{2\pi}{3}$ (d) None of these

57. If $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ and $A^8=aA+bI$, then $(a,b)=$

(a) (8, 7) (b) (–7, 8) (c) (8, – 7) (d) (–8, – 7)

58. $\int_1^e \log x\,dx=$

(a) 1 (b) $e-1$ (c) $e+1$ (d) 0

59. The shortest distance between the lines $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ and $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ is

(a) $\sqrt{30}$ (b) $2\sqrt{30}$ (c) $5\sqrt{30}$ (d) $3\sqrt{30}$

60. If in the expansion of $(1+px)^n, n\in N$, the coefficient of x and x^2 are 8 and 24, then

(a) $n=3, p=2$ (b) $n=5, p=3$
(c) $n=4, p=3$ (d) $n=4, p=2$

SOLUTIONS

1. **(d)**

2. **(None of the options is correct):** We have,

$$\lim_{x\to\infty}\left(\sqrt{a^2x^2+bx+x}-ax\right)$$

$$=\lim_{x\to\infty}\left(\sqrt{a^2x^2+bx+x}-ax\times\frac{\sqrt{a^2x^2+bx+x}+ax}{\sqrt{a^2x^2+bx+x}+ax}\right)$$

$$=\lim_{x\to\infty}\left(\frac{(a^2x^2+bx+x)-a^2x^2}{x\left\{\sqrt{a^2+\frac{b}{x}+\frac{1}{x}}+a\right\}}\right)$$

$$=\lim_{x\to\infty}\left(\frac{(b+1)x}{x\left\{\sqrt{a^2+\frac{b}{x}+\frac{1}{x}}+a\right\}}\right)$$

$$=\frac{b+1}{2a}$$

3. **(b) :** Given inequality is,

$$\frac{6x}{4x-1}<\frac{1}{2}\Rightarrow\frac{12x}{4x-1}<1$$

$$\Rightarrow\frac{12x}{4x-1}-1<0\Rightarrow\frac{8x+1}{4x-1}<0 \quad ...(i)$$

For above inequality (i), we have two cases either (I) $8x+1>0, 4x-1<0$
or (II) $8x+1<0, 4x-1>0$

From (I), we get $x>\frac{-1}{8}, x<\frac{1}{4}$

$$\therefore\quad x\in\left(-\frac{1}{8},\frac{1}{4}\right) \quad ...(ii)$$

From (II), $x<\frac{-1}{8}, x>\frac{1}{4}$, both of which cannot be true

$\therefore$ (ii) represent the solution set of given inequality.

4. **(c)**

5. **(a) :** We have, $BA=B$

$\Rightarrow (BA)B=BB\Rightarrow B(AB)=B^2$

$\Rightarrow BA=B^2\Rightarrow B=B^2$

6. **(a) :** To form a triangle, 3 line segments are needed.

We can choose 3 line segments from 5 line segments in ${}^5C_3$ ways.

We know that in a triangle, sum of two sides is always greater than third side.

So, line segments of length (2, 3, 5), (2, 3, 6) and (2, 4, 6) do not form any triangle.

$\therefore$ Total number of triangles formed $={}^5C_3-3$

7. **(b) :** Number of numbers formed $=\frac{6\times 6!}{2!\times 3!}$

$=360$

8. (b) : Let E_1, E_2 and E_3 be the events such that drawn ball is white, blue and black respectively.

$\therefore$ The required probability

$$= {}^3P_1 P(E_1)P(E_1)P(\bar{E}_1) + {}^3P_1 P(E_2)P(E_2)P(\bar{E}_2) + {}^3P_1 P(E_3)P(E_3)P(\bar{E}_3)$$

$$= 3\times\frac{2}{9}\times\frac{1}{8}\times\frac{7}{7} + 3\times\frac{3}{9}\times\frac{2}{8}\times\frac{6}{7} + 3\times\frac{4}{9}\times\frac{3}{8}\times\frac{5}{7}$$

$$= 3\left[\frac{14+36+60}{9\times8\times7}\right] = \frac{3\times110}{9\times8\times7} = \frac{55}{84}$$

9. (c) : 6 boys and 6 girls sit in a row in 12! ways.

Number of ways when all six girls sit together $= 7! \times 6!$

$\therefore$ Required probability $= \dfrac{7!\times6!}{12!} = \dfrac{1}{132}$

10. (c)

11. (c) : Total number of balls = 9

Probability of drawn two white balls $= \dfrac{3}{9}\times\dfrac{2}{8} = \dfrac{1}{12}$

12. (b)

13. (b) : We have, $f(x) = e^x$ and $g(x) = \log e^x$

Now, $f(g(x)) = f(\log e^x) = e^{\log ex} = e^x$

and $g(f(x)) = g(e^x) = \log e^{ex} = e^x \log e = e^x$

Hence, $f(g(x)) = g(f(x))$

14. (b) : Since number of one-one onto functions from a set A having n elements to itself is $n!$.

15. (c) : We have, $y = |\cos x| + |\sin x|$

At $x = \dfrac{2\pi}{3}$, $\cos x$ is – ve and $\sin x$ is +ve

$\therefore$ $y = -\cos x + \sin x$

$\Rightarrow$ $\dfrac{dy}{dx} = \sin x + \cos x$

$\therefore$ $\left.\dfrac{dy}{dx}\right|_{\text{at } x=2\pi/3} = \sin\left(\dfrac{2\pi}{3}\right) + \cos\left(\dfrac{2\pi}{3}\right) = \dfrac{\sqrt{3}}{2} - \dfrac{1}{2} = \dfrac{\sqrt{3}-1}{2}$

16. (a)

17. (c) : Since $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix} \Rightarrow |A| = a\cdot a\cdot a = a^3$

Using formula $|\text{adj } A| = |A|^{n-1}$, we get

$\det(\text{adj } A) = (a^3)^{3-1} = (a^3)^2 = a^6$

18. (a)

19. (b) : We have, $\dfrac{dy}{dx} = \dfrac{y + x\tan\dfrac{y}{x}}{x}$ ...(i)

Given differential equation is in homogeneous form

$\therefore$ Put $y = vx$ in (i), we get

$$v + x\frac{dv}{dx} = v + \tan v \Rightarrow \frac{1}{\tan v}dv = \frac{dx}{x}$$

Taking integration on both sides, we get

$$\log(\sin v) = \log x + \log c$$

$$\Rightarrow \log\frac{\sin v}{x} = \log c \Rightarrow \sin\left(\frac{y}{x}\right) = xc$$

20. (b) : Given D.E. is

$$\frac{dy}{dx} + y\cdot g'(x) = g(x)\cdot g'(x)$$

This is in the form of linear differential equation

So, I.F. $= e^{\int g'(x)dx} = e^{g(x)}$

Required solution is

$$y\cdot e^{g(x)} = \int g(x)\cdot g'(x)\cdot e^{g(x)}\,dx + C$$

Put $g(x) = t \Rightarrow g'(x)\,dx = dt$

$\therefore$ $y\cdot e^{g(x)} = \int te^t\,dt + C'$

$\Rightarrow$ $y\cdot e^{g(x)} = te^t - e^t + C'$

$\Rightarrow$ $y\cdot e^{g(x)} = (g(x) - 1)e^{g(x)} + C'$

or $(y - g(x) + 1)e^{g(x)} = C'$

Taking log on both sides, we get

$$g(x) + \log(1 + y - g(x)) = C$$

21. (a) **22. (c)** **23. (c)**

24. (b) : Let $I = \displaystyle\int\frac{x^3-1}{x^3+x}dx = \int\left(1 - \frac{x+1}{x^3+x}\right)dx$

$$= \int 1dx - \int\frac{x+1}{x(x^2+1)}dx = x - \int\frac{x+1}{x(x^2+1)}dx \quad \text{...(i)}$$

Now $\dfrac{x+1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

(By using partial fractions)

$\Rightarrow$ $x + 1 = A(x^2 + 1) + (Bx + C)x$

$\Rightarrow$ $x + 1 = (A + B)x^2 + Cx + A$

Comparing coefficients of x^2, x and constant, we get

$A + B = 0$, $C = 1$, $A = 1$

$\Rightarrow$ $B = -1$

$\therefore$ From (i), we get

$$I = x - \int\frac{1}{x}dx - \int\frac{1-x}{x^2+1}dx$$

$$= x - \log x - \int\frac{1}{x^2+1}dx + \frac{1}{2}\int\frac{2x}{x^2+1}dx$$

$$= x - \log x - \tan^{-1}x + \frac{1}{2}\log(x^2+1) + c$$

25. (c) : L.H.S $= \displaystyle\int\frac{\cos 8x + 1}{\tan 2x - \cot 2x}dx$

$$= \int\frac{2\cos^2 4x}{\left(\dfrac{\sin^2 2x - \cos^2 2x}{\sin 2x\cos 2x}\right)}dx$$

$$= -\int\frac{\cos^2 4x(2\sin 2x\cos 2x)}{(\cos^2 2x - \sin^2 2x)}dx$$

$$= -\int \frac{\cos^2 4x \times \sin 4x}{\cos 4x}\, dx$$

$$= -\frac{1}{2}\int 2\sin 4x \cos 4x \; dx$$

$$= -\frac{1}{2}\int \sin 8x \, dx = \frac{1}{2}\times\frac{\cos 8x}{8} + c$$

Now, $-\frac{1}{2}\frac{\cos 8x}{8} + c = a\cos 8x + c \;\; \therefore \;\; a = \frac{1}{16}$

26. (d) : We have, $\int_0^{\pi} x\, f(\sin x)dx = A\int_0^{\pi/2} f(\sin x)dx$

Let $I = \int_0^{\pi} x\cdot f(\sin x)\, dx$ ...(i)

$\Rightarrow \quad I = \int_0^{\pi} (\pi - x)\, f(\sin(\pi - x))\, dx$

$\Rightarrow \quad I = \int_0^{\pi} (\pi - x)\, f(\sin x)dx$ ...(ii)

Adding (i) and (ii), we get

$$2I = \pi\int_0^{\pi} f(\sin x)dx \;\Rightarrow\; I = \frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

$$\Rightarrow\; I = \frac{\pi}{2}\times 2\int_0^{\pi/2} f(\sin x)dx \;\Rightarrow\; I = \pi\int_0^{\pi/2} f(\sin x)dx.$$

$\therefore$ On comparing, we get $A = \pi$

27. (c) : We have, $\cot^{-1}(21) + \cot^{-1}(13) + \cot^{-1}(-8)$

$$= \tan^{-1}\left(\frac{1}{21}\right) + \tan^{-1}\left(\frac{1}{13}\right) + \cot^{-1}(-8)$$

$$= \tan^{-1}\left(\frac{\frac{1}{21}+\frac{1}{13}}{1-\frac{1}{21}\times\frac{1}{13}}\right) + \cot^{-1}(-8)$$

$$= \tan^{-1}\left(\frac{34}{272}\right) + (\pi - \cot^{-1}8) \;= \tan^{-1}\left(\frac{1}{8}\right) - \cot^{-1}8 + \pi$$

$$= \tan^{-1}\left(\frac{1}{8}\right) - \tan^{-1}\left(\frac{1}{8}\right) + \pi \;= \pi$$

28. (d) : $\tan\left(\cos^{-1}\left(\frac{1}{5\sqrt{2}}\right) - \sin^{-1}\left(\frac{4}{\sqrt{17}}\right)\right)$

$$= \tan\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{1}{5\sqrt{2}}\right) - \sin^{-1}\left(\frac{4}{\sqrt{17}}\right)\right)$$

$$= \tan\left(\frac{\pi}{2} - \left(\sin^{-1}\frac{4}{\sqrt{17}} + \sin^{-1}\frac{1}{5\sqrt{2}}\right)\right)$$

$$= \tan\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{4}{\sqrt{17}}\cdot\sqrt{1-\frac{1}{50}} + \frac{1}{5\sqrt{2}}\cdot\sqrt{1-\frac{16}{17}}\right)\right)$$

$$= \tan\left(\frac{\pi}{2} - \sin^{-1}\left(\frac{4}{\sqrt{17}}\cdot\frac{7}{5\sqrt{2}} + \frac{1}{5\sqrt{2}}\cdot\frac{1}{\sqrt{17}}\right)\right)$$

$$= \cot\left(\sin^{-1}\frac{29}{5\sqrt{2}\sqrt{17}}\right) = \cot\left(\cot^{-1}\frac{3}{29}\right) = \frac{3}{29}.$$

29. (b) : We have, $11^3 + 12^3 + 13^3 + ...+20^3$

$= (1^3 + 2^3 + ... + 10^3 + 11^3 + 12^3 + ... + 20^3)$

$\qquad - (1^3 + 2^3 + ... + 10^3)$

$$= \Sigma(20)^3 - \Sigma(10)^3 = \left(\frac{20(20+1)}{2}\right)^2 - \left(\frac{10(10+1)}{2}\right)^2$$

$= (10 \times 21)^2 - (5 \times 11)^2 = (210 - 55)\,(210 + 55)$

$= 155 \times 265 = 41075 = 5 \times 8215$

30. (c)

31. (c) : $n(A) = 4$ (given)

$\therefore$ Total no. of subsets of $A = 2^4 = 16$

$\therefore$ No. of proper subsets $= 16 - 1 = 15$

32. (d) **33. (d)**

34. (c) : Given equation of lines are

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ...(i)$$

and $\frac{x}{b} + \frac{y}{a} = 1$ ...(ii)

$\Rightarrow \quad bx + ay = ab$ ...(iii)

and $ax + by = ab$ ...(iv)

Solving (iii) and (iv), we get

$$(a^2 - b^2)y = a^2b - ab^2 = ab(a-b) \Rightarrow \; y = \frac{ab}{a+b}$$

Substituting the value of y in (iii), we get

$$bx + a\cdot\frac{ab}{a+b} = ab \;\Rightarrow\; bx = ab - \frac{a^2b}{a+b}$$

$$\Rightarrow \quad bx = \frac{ab^2}{a+b} \Rightarrow x = \frac{ab}{a+b}$$

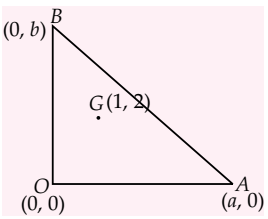
$\therefore$ Point of intersection is $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$

Since equation of the line passing through origin is $y = mx$

$\therefore$ When it pass through $\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$ then, we get $m = 1$

Hence, required equation of line is, $y - x = 0$

35. (b) : Since straight line meets the coordinate axes at A and B, so equation of line in intercept form is $\frac{x}{a} + \frac{y}{b} = 1$



$\therefore \quad G\left(\frac{0+a+0}{3}, \frac{0+0+b}{3}\right) = (1,2)$ (given)

$\Rightarrow \quad \frac{a}{3} = 1 \Rightarrow a = 3, \quad \frac{b}{3} = 2 \Rightarrow b = 6$

Hence, required equation of line is

$$\frac{x}{3} + \frac{y}{6} = 1 \Rightarrow 2x + y = 6$$

36. (a) : Let d.r's of line are $< a, b, c >$, then

$2a - 3b + c = 0$ and $a + 2b - 2c = 0$

$\Rightarrow \quad \frac{a}{6-2} = \frac{b}{1+4} = \frac{c}{4+3} \Rightarrow \frac{a}{4} = \frac{b}{5} = \frac{c}{7}$

$\therefore$ d.r's of line are $< 4, 5, 7 >$.

37. (a)

38. (b) : (i) When $x \in \left[0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right)$ then $\cot x \geq 0$

$\Rightarrow \quad \cot x = \cot x + \frac{1}{\sin x} \Rightarrow \frac{1}{\sin x} = 0$

$\Rightarrow$ No solution exist.

(ii) When $x \in \left(\frac{\pi}{2}, \pi\right) \cup \left(\frac{3\pi}{2}, 2\pi\right)$ then $\cot x < 0$

$\therefore \quad -\cot x = \cot x + \frac{1}{\sin x}$

$\Rightarrow \quad -2\cot x = \frac{1}{\sin x} \Rightarrow \cos x = \frac{-1}{2} \Rightarrow x = \frac{2\pi}{3}$

39. (d) : $\frac{\sin x - \sin 3x}{\sin^2 x - \cos^2 x} = \frac{2\cos 2x \sin(-x)}{-\cos 2x} = 2 \sin x$

40. (b) : cosec A (sin B cos C + sin B sin C)

= cosec A sin $(B + C)$

= cosec A sin $(180° - A)$

[$\because A$, B and C are angles of a triangle]

$= \frac{1}{\sin A} \sin A = 1$

41. (c)

42. (a) : According to question,

$\vec{a} \cdot (\vec{b} + \vec{c}) = 0, \quad \vec{b} \cdot (\vec{c} + \vec{a}) = 0, \quad \vec{c} \cdot (\vec{a} + \vec{b}) = 0$

$\Rightarrow \quad \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c} = 0, \quad$ ...(i)

$\vec{b} \cdot \vec{c} + \vec{b} \cdot \vec{a} = 0, \quad$ ...(ii)

$\vec{c} \cdot \vec{a} + \vec{c} \cdot \vec{b} = 0 \quad$ ...(iii)

Adding (i), (ii) and (iii), we get

$2(\vec{a} \cdot \vec{b} + \vec{b} \cdot \vec{c} + \vec{c} \cdot \vec{a}) = 0 \quad$ ..(iv)

Now, $\left|\vec{a} + \vec{b} + \vec{c}\right|^2 = |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a})$

$= |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2$ (Using (iv))

43. (d)

44. (d) : $S^2 = at^2 + 2bt + c \quad$ ...(i)

Differentiating (i) w.r.t 't', we get

$$2S\frac{dS}{dt} = 2at + 2b \Rightarrow \frac{dS}{dt} = \left(\frac{at+b}{S}\right) \quad \text{...(ii)}$$

Differentiating (ii) w.r.t. 't', we get

$$\frac{d^2S}{dt^2} = \frac{S\frac{d}{dt}(at+b) - (at+b)\frac{dS}{dt}}{S^2}$$

$$\Rightarrow \quad \frac{d^2S}{dt^2} = \frac{aS - (at+b)\frac{(at+b)}{S}}{S^2} \quad \text{(Using (ii))}$$

$$\Rightarrow \quad \frac{d^2S}{dt^2} = \frac{aS^2 - (at+b)^2}{S^3}$$

$$\Rightarrow \quad \frac{d^2S}{dt^2} = \frac{a(at^2 + 2bt + c) - \left[(at)^2 + b^2 + 2atb\right]}{S^3} \quad \text{[Using (i)]}$$

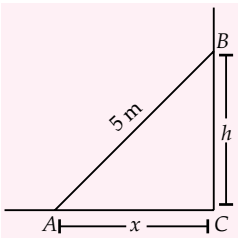
$$\Rightarrow \quad \frac{d^2S}{dt^2} = \frac{a^2t^2 + 2abt + ac - a^2t^2 - b^2 - 2atb}{S^3}$$

$$\Rightarrow \quad \frac{d^2S}{dt^2} = \frac{ac - b^2}{S^3} \Rightarrow \text{Acceleration} = \left(\frac{d^2S}{dt^2}\right) \propto \frac{1}{S^3}$$

45. (b) : Let AB be the ladder of length 5 m.

We are given,

$\frac{dx}{dt} = 2$ m/sec.



In ΔABC,

$AB^2 = AC^2 + BC^2$

$\Rightarrow \quad (5)^2 = x^2 + h^2 \quad$ ...(i)

Differentiating (i) w.r.t. t, we get

$$0 = 2x\frac{dx}{dt} + 2h\frac{dh}{dt} \Rightarrow \frac{dh}{dt} = \frac{-x}{h}\frac{dx}{dt} \quad \text{...(ii)}$$

Now, from (i), when $x = 4$

$h^2 = 25 - 16 \Rightarrow h^2 = 9 \Rightarrow h = 3$ m

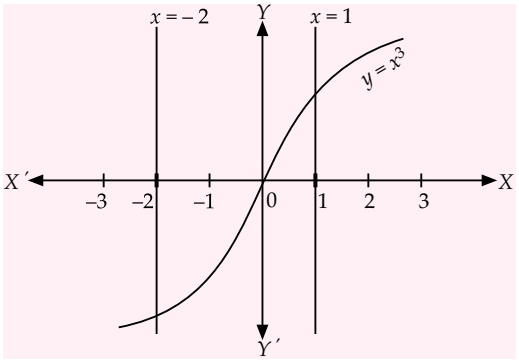
From (ii)

$$\left[\frac{dh}{dt}\right] = \frac{-4}{3} \times 2 = \frac{-8}{3}$$

$\because$ The negative sign shows the height decreases and decreasing rate is $\frac{8}{3}$ m/sec.

46. (b)

47. (d) :



Required Area $= \left|\int_{-2}^{0} x^3 dx\right| + \int_{0}^{1} x^3 dx$

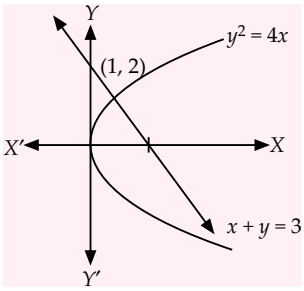
$= \left|\left[\frac{x^4}{4}\right]_{-2}^{0}\right| + \left[\frac{x^4}{4}\right]_{0}^{1} = \frac{16}{4} + \frac{1}{4} = \frac{17}{4}$ sq. units

48. (a) : We have, $y^2 = 4x$

Differentiating w.r.t. 'x', we get

$2y\frac{dy}{dx} = 4 \Rightarrow \frac{dy}{dx} = \frac{2}{y}$

$\left(\frac{dy}{dx}\right)_{(1,2)} = \frac{2}{2} = 1$



Equation of normal to the curve at (1, 2) is

$y - y_1 = \frac{1}{\left(-\frac{dy}{dx}\right)}(x - x_1)$

$\Rightarrow \quad (y-2) = -\frac{1}{1}(x-1) \Rightarrow y - 2 = -x + 1 \Rightarrow x + y = 3$

The line $x + y = 3$ meets the x-axis at $x = 3$

$\therefore$ Required Area $= \int_0^1 \sqrt{4x}\, dx + \int_1^3 (3-x)\, dx$

$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^1 + \left[3x - \frac{x^2}{2}\right]_1^3 = \frac{4}{3}(1) + \left[9 - \frac{9}{2} - 3 + \frac{1}{2}\right]$

$= \frac{4}{3} + \left(\frac{9}{2} - \frac{5}{2}\right) = \frac{4}{3} + \frac{4}{2} = \frac{4}{3} + 2 = \frac{10}{3}$ sq. units

49. (a) **50. (c)**

51. (d): $\frac{1}{\sqrt{(3x+1)}}\left\{\left(\frac{1+\sqrt{3x+1}}{2}\right)^7 - \left(\frac{1-\sqrt{3x+1}}{2}\right)^7\right\}$

$= \frac{1}{2^7\sqrt{(3x+1)}}\left\{\left(1+\sqrt{3x+1}\right)^7 - \left(1-\sqrt{3x+1}\right)^7\right\}$ ...(i)

Now,

$\left(1+\sqrt{3x+1}\right)^7 - \left(1-\sqrt{3x+1}\right)^7$

$= \left[{}^7C_0 + {}^7C_1\sqrt{(3x+1)} + {}^7C_2\left(\sqrt{3x+1}\right)^2 + \ldots + {}^7C_7\left(\sqrt{3x+1}\right)^7\right]$

$-\left[{}^7C_0 - {}^7C_1\sqrt{3x+1} + {}^7C_2\left(\sqrt{(3x+1)}\right)^2 \ldots - {}^7C_7\left(\sqrt{3x+1}\right)^7\right]$

$= 2\left[{}^7C_1\left(\sqrt{3x+1}\right) + {}^7C_3\left(\sqrt{3x+1}\right)^3 + {}^7C_5\left(\sqrt{3x+1}\right)^5 + {}^7C_7\left(\sqrt{3x+1}\right)^7\right]$

$= 2\sqrt{3x+1} \times \left[7 + 35(3x+1) + 21(3x+1)^2 + (3x+1)^3\right]$

Now, putting above value in (i), so the given expression becomes $\frac{1}{2^6}[42 + 105x + 21(3x+1)^2 + (3x+1)^3]$

$\therefore$ Degree of a polynomial is the highest power of x. So, degree of given expression is 3.

52. (c) : Let $z = i \sin\frac{\pi}{19}$ or $z = 0 + i\sin\frac{\pi}{19}$

Let amplitude of z be θ then

$\sin\theta = \frac{\sin\frac{\pi}{19}}{\sqrt{0^2 + \left(\sin\frac{\pi}{19}\right)^2}} \quad \left[\sin\theta = \frac{b}{\sqrt{a^2+b^2}}\right]$

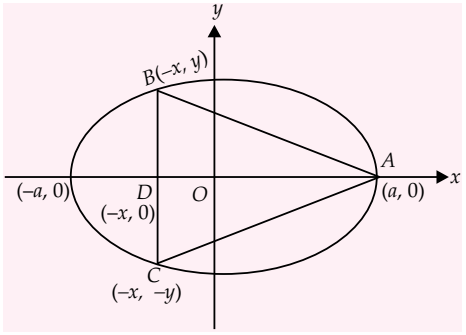
or $\sin\theta = \frac{\sin\frac{\pi}{19}}{\sin\frac{\pi}{19}} = 1 \Rightarrow \theta = \frac{\pi}{2}$

53. (b) : Let ΔABC be an isosceles triangle inscribed in the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Let co-ordinate of A be $(a, 0)$

Area of ΔABC is

$A = \frac{1}{2} \times AD \times BC = \frac{1}{2} \times (a+x) \times 2y = y(a+x)$

$= \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right)}(a+x) = \frac{b}{a}\sqrt{a^2 - x^2}\,(a+x)$



$\Rightarrow \quad \frac{dA}{dx} = \frac{b}{a}\left[(a+x).\frac{1}{2} \times \frac{-2x}{\sqrt{a^2-x^2}} + \sqrt{a^2-x^2}\right]$

$= \frac{b}{a}\left[\frac{-x(a+x) + (a^2-x^2)}{\sqrt{a^2-x^2}}\right]$

Put $\frac{dA}{dx} = 0 \Rightarrow 2x^2 + ax - a^2 = 0 \Rightarrow 2x^2 + 2ax - ax - a^2 = 0$

$\Rightarrow \quad (2x-a)(x+a) = 0 \Rightarrow x = \frac{a}{2}, -a$

Since $\frac{d^2A}{dx^2} < 0$ for $x = \frac{a}{2}$

$\therefore$ Max. Area $= \frac{b}{a}.\left(a + \frac{a}{2}\right)\sqrt{a^2 - \frac{a^2}{4}}$

$= \frac{b}{a}.\frac{3}{2}a.\frac{\sqrt{3}a}{2} = \frac{3\sqrt{3}ab}{4}$ sq. units

54. (b) : $f(x+y)=f(x)\,f(y)$

Putting $x=y=0$

$\Rightarrow \quad f(0+0)=f(0)\,f(0) \Rightarrow f(0)=[f(0)]^2 \Rightarrow f(0)=1$

Now,

$$f'(5)=\lim_{h\to 0}\frac{f(5+h)-f(5)}{h}=\lim_{h\to 0}\frac{f(5)f(h)-f(5)}{h}$$

$$=\lim_{h\to 0}f(5)\left[\frac{f(h)-1}{h}\right]$$

$$=f(5)\lim_{h\to 0}\left[\frac{f(h)-f(0)}{h}\right] \qquad [\because\ f(0)=1]$$

$$=2\times f'(0)=2\times 3=6$$

55. (d) : $f(x)=\log_{x2}(\log x)$

$$\Rightarrow \quad f(x)=\frac{1}{2}\left[\log_x(\log x)\right] \left[\because\ \log_{a^n}b=\frac{1}{n}\log_a b\right]$$

$$\Rightarrow \quad f(x)=\frac{1}{2}\left[\frac{\log(\log x)}{\log x}\right] \left(\because\ \log_a b=\frac{\log b}{\log a}\right)$$

$$\Rightarrow \quad f'(x)=\frac{1}{2}\left[\frac{\log(x)\frac{d}{dx}[\log(\log(x))]-\log(\log x)\frac{d}{dx}\log x}{(\log x)^2}\right]$$

$$=\frac{1}{2}\left[\frac{\log(x)\frac{1}{\log x}\times\frac{1}{x}-\log(\log x)\frac{1}{x}}{(\log x)^2}\right]=\frac{1}{2}\left[\frac{\frac{1}{x}-\frac{\log(\log(x))}{x}}{(\log x)^2}\right]$$

$$\Rightarrow \quad f'(x)=\frac{1}{2}\left[\frac{1-\log\log(x)}{x(\log x)^2}\right]$$

$$\Rightarrow \quad f'(e)=\frac{1}{2}\left[\frac{1-\log\log(e)}{e(\log e)^2}\right]=\frac{1}{2}\left[\frac{1-0}{e}\right]=\frac{1}{2e}$$

56. (b) : $f(x)=\log(\sin(x))$ is continuous in $\left[\frac{\pi}{6},\frac{5\pi}{6}\right]$ and differentiable in $\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ as its derivative $f'(x)=\frac{1}{\sin x}\cos x=\cot x$

Now, $f\left(\frac{\pi}{6}\right)=\log\left(\sin\left(\frac{\pi}{6}\right)\right)=\log\left(\frac{1}{2}\right)$

$$f\left(\frac{5\pi}{6}\right)=\log\sin\left(\frac{5\pi}{6}\right)=\log\sin\left(\pi-\frac{\pi}{6}\right)=\log\left(\frac{1}{2}\right)$$

Now, according to Lagrange's theorem, there exist a point $c\in\left(\frac{\pi}{6},\frac{5\pi}{6}\right)$ such that

$$f'(c)=\frac{f\left(\frac{\pi}{6}\right)-f\left(\frac{5\pi}{6}\right)}{\left(\frac{\pi}{6}-\frac{5\pi}{6}\right)} \Rightarrow \cot c=0 \Rightarrow c=\frac{\pi}{2}$$

57. (c) : We have, $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$

$$A^2=A\cdot A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}$$

$$A^3=A^2\cdot A=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}=\begin{bmatrix}1 & 0\\ 3 & 1\end{bmatrix}$$

Similarly, $A^8=\begin{bmatrix}1 & 0\\ 8 & 1\end{bmatrix}$

Now, $A^8=aA+bI$

$$\Rightarrow \quad \begin{bmatrix}1 & 0\\ 8 & 1\end{bmatrix}=a\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}+b\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix}1 & 0\\ 8 & 1\end{bmatrix}=\begin{bmatrix}a+b & 0\\ a & a+b\end{bmatrix}$$

$\therefore \quad a=8$ and $a+b=1 \Rightarrow b=1-8=-7$

58. (a) : We have, $\int_1^e \log x\,dx$

$$=\left[\log x\int 1\,dx\right]_1^e-\int_1^e\left(\frac{d}{dx}\log x\int 1dx\right)dx$$

$$=\left[\log x\cdot x\right]_1^e-\int_1^e\frac{1}{x}\cdot x\,dx$$

$$=\left[e\log e-1\log(1)\right]-\int_1^e 1\,dx=e-[x]_1^e=e-[e-1]=1$$

59. (d) **60. (d)**

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KCET

SOLVED PAPER 2015

1. A man takes a step forward with probability 0.4 and one step backward with probability 0.6, then the probability that at the end of eleven steps he is one step away from the starting point, is

(a) ${}^{11}C_6 \times (0.24)^5$ (b) ${}^{11}C_6 \times (0.72)^6$

(c) ${}^{11}C_5 \times (0.48)^5$ (d) ${}^{11}C_5 \times (0.12)^5$

2. $\int_0^{\pi/4} \log\left(\frac{\sin x + \cos x}{\cos x}\right) dx$ is equal to

(a) $\frac{\pi}{2}\log 2$ (b) $\log 2$ (c) $\frac{\pi}{4}\log 2$ (d) $\frac{\pi}{8}\log 2$

3. Area bounded by $y = x^3$, $y = 8$ and $x = 0$ is _____.

(a) 14 sq. units (b) 6 sq. units

(c) 2 sq. units (d) 12 sq. units

4. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} + \hat{j} - \hat{k}$, a vector in the plane of $\vec{a}$ and $\vec{b}$ whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$ is _____.

(a) $4\hat{i} + \hat{j} - 4\hat{k}$ (b) $4\hat{i} - \hat{j} + 4\hat{k}$

(c) $3\hat{i} + \hat{j} - 3\hat{k}$ (d) $\hat{i} + \hat{j} - 2\hat{k}$

5. The mean deviation from the data 3, 10, 10, 4, 7, 10, 5 is

(a) 2 (b) 2.57 (c) 3 (d) 3.75

6. The probability distribution of x is

x	0	1	2	3
$P(x)$	0.2	k	k	$2k$

Find the value of k.

(a) 0.3 (b) 0.1 (c) 0.2 (d) 0.4

7. If the function $g(x)$ is defined by $g(x) = \frac{x^{200}}{200} + \frac{x^{199}}{199} + \frac{x^{198}}{198} + \ldots\ldots + \frac{x^2}{2} + x + 5$, then $g'(0)$ = _____.

(a) 200 (b) 5 (c) 1 (d) 100

8. A box contains 6 red marbles numbers from 1 through 6 and 4 white marbles 12 through 15. Find the probability that a marble drawn 'at random' is white and odd numbered.

(a) $\frac{1}{5}$ (b) $\frac{1}{6}$ (c) 5 (d) 6

9. $\lim_{x \to 0} \frac{1 - \cos x}{x^2}$ is _____.

(a) 3 (b) $\frac{1}{3}$ (c) 2 (d) $\frac{1}{2}$

10. $f(x) = \begin{cases} 3x - 8 & \text{if } x \le 5 \\ 2k & \text{if } x > 5 \end{cases}$ is continuous, find k.

(a) $\frac{3}{7}$ (b) $\frac{7}{2}$ (c) $\frac{2}{7}$ (d) $\frac{4}{7}$

11. If $f(x) = 2x^2$, find $\frac{f(3.8) - f(4)}{3.8 - 4}$.

(a) 156 (b) 0.156 (c) 1.56 (d) 15.6

12. If $x = ct$ and $y = \frac{c}{t}$, find $\frac{dy}{dx}$ at $t = 2$.

(a) 4 (b) 0 (c) $\frac{1}{4}$ (d) $-\frac{1}{4}$

13. A balloon which always remains spherical is being inflated by pumping in 10 cubic centimeters of gas per second. Find the rate at which the radius of the balloon is increasing when the radius is 15 cm.

(a) $\frac{1}{9\pi}$ cm/sec (b) $\frac{1}{\pi}$ cm/sec

(c) $\frac{1}{90\pi}$ cm/sec (d) $\frac{1}{30\pi}$ cm/sec

14. $\int \frac{\sin^2 x}{1 + \cos x} dx$ is equal to

(a) $x - \sin x + C$ (b) $\cos x + C$

(c) $x + \sin x + C$ (d) $\sin x + C$

15. $\int e^x \left(\frac{1+\sin x}{1+\cos x}\right) dx$ is _____ .

(a) $\tan\left(\frac{x}{2}\right)+C$ (b) $e^x \sin x + C$

(c) $e^x \tan\left(\frac{x}{2}\right)+C$ (d) $e^x + C$

16. If 1, ω, ω^2 are three cube roots of unity, then $(1 - \omega + \omega^2)(1 + \omega - \omega^2)$ is _____

(a) 2 (b) 4 (c) 1 (d) 3

17. Solve for x

$$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1} x,\ x>0$$

(a) 1 (b) $\frac{1}{\sqrt{3}}$ (c) $\sqrt{3}$ (d) -1

18. The system of linear equations $x + y + z = 6$, $x + 2y + 3z = 10$ and $x + 2y + az = b$ has no solution when _____

(a) $a = 3, b \neq 10$ (b) $b = 3, a \neq 10$

(c) $a = 2, b \neq 3$ (d) $b = 2, a = 3$

19. The value of $\tan(1°) + \tan(89°)$ is _____

(a) $\frac{2}{\sin(2°)}$ (b) $\frac{1}{\sin(2°)}$

(c) $\frac{1}{\sin(1°)}$ (d) $\frac{2}{\sin(1°)}$

20. If $\frac{(x+1)^2}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$, then

$$\text{cosec}^{-1}\left(\frac{1}{A}\right)+\cot^{-1}\left(\frac{1}{B}\right)+\sec^{-1} C = \_\_\_\_\_$$

(a) 0 (b) $\frac{\pi}{2}$ (c) $\frac{5\pi}{6}$ (d) $\frac{\pi}{6}$

21. The remainder obtained when $1! + 2! + 3! + ... + 11!$ is divided by 12 is _____ .

(a) 8 (b) 6 (c) 9 (d) 7

22. If $\alpha \leq 2 \sin^{-1} x + \cos^{-1} x \leq \beta$, then

(a) $\alpha=\frac{-\pi}{2},\ \beta=\frac{3\pi}{2}$ (b) $\alpha = 0,\ \beta = 2\pi$

(c) $\alpha=\frac{-\pi}{2},\ \beta=\frac{\pi}{2}$ (d) $\alpha = 0,\ \beta = \pi$

23. If $A=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$, then A^2 is equal to _____

(a) $\begin{bmatrix}1 & 0\\ 1 & 0\end{bmatrix}$ (b) $\begin{bmatrix}0 & 1\\ 0 & 1\end{bmatrix}$ (c) $\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$ (d) $\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

24. The function $f(x) = [x]$, where $[x]$ denotes greatest integer function is continuous at _____

(a) -2 (b) 1.5 (c) 4 (d) 1

25. If $y=\log\left(\frac{1-x^2}{1+x^2}\right)$, then $\frac{dy}{dx}$ is equal to _____

(a) $\frac{4x^3}{1-x^4}$ (b) $\frac{-4x^3}{1-x^4}$ (c) $\frac{-4x}{1-x^4}$ (d) $\frac{1}{4-x^4}$

26. The two curves $x^3 - 3xy^2 + 2 = 0$ and $3x^2y - y^3 = 2$

(a) cut at right angle (b) cut at angle $\frac{\pi}{4}$

(c) touch each other (d) cut at angle $\frac{\pi}{3}$

27. If x is real, then the minimum value of $x^2 - 8x + 17$ is _____

(a) 2 (b) 4 (c) 1 (d) 3

28. $\int_{-\pi/4}^{\pi/4} \frac{dx}{1+\cos 2x}$ is equal to

(a) 1 (b) 0 (c) 2 (d) 4

29. The order of differential equation of all circles of given radius 'a' is _____

(a) 2 (b) 3 (c) 4 (d) 1

30. The solution of differential equation

$x\frac{dy}{dx}+2y=x^2$ is _____

(a) $y=\frac{x^2}{4}+C$ (b) $y=\frac{x^4+C}{4x^2}$

(c) $y=\frac{x^2+C}{4x^2}$ (d) $y=\frac{x^4+C}{x^2}$

31. If $\sin x + \sin y = \frac{1}{2}$ and $\cos x + \cos y = 1$, then $\tan(x + y)$ = _____

(a) $-\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{8}{3}$ (d) $\frac{-8}{3}$

32. If $A=\begin{bmatrix}\alpha & 2\\ 2 & \alpha\end{bmatrix}$ and $|A^3| = 27$, then α = _____

(a) ± 2 (b) $\pm\sqrt{5}$ (c) ± 1 (d) $\pm\sqrt{7}$

33. If $P=\begin{vmatrix}x & 1\\ 1 & x\end{vmatrix}$ and $Q=\begin{vmatrix}x & 1 & 1\\ 1 & x & 1\\ 1 & 1 & x\end{vmatrix}$, then $\frac{dQ}{dx}$ = _____.

(a) $1 - 3P$ (b) $3P$ (c) $3P + 1$ (d) $-3P$

34. A line passes through (2, 2) and is perpendicular to the line $3x + y = 3$. Its y-intercept is _____ .

(a) $\frac{2}{3}$ (b) 1 (c) $\frac{1}{3}$ (d) $\frac{4}{3}$

35. Let $f : R \to R$ be defined by $f(x) = \frac{1}{x} \forall x \in R$, then f is _____.

(a) onto (b) not defined
(c) one-one (d) bijective

36. The solution set of the inequation $\frac{x^2+6x-7}{|x+4|} < 0$ is _____

(a) $(-7, -4)$ (b) $(-7, -4) \cup (4, 1)$
(c) $(-7, 1)$ (d) $(-7, -4) \cup (-4, 1)$

37. $f(x) = \frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)$, $-1 < x < 1$ and $g(x) = \sqrt{(3+4x-4x^2)}$. Find the domain of $(f+g)$.

(a) $\left(\frac{-1}{2}, 1\right]$ (b) $(-1, 1)$

(c) $\left[\frac{-1}{2}, 1\right)$ (d) $\left[-\frac{1}{2}, \frac{3}{2}\right]$

38. Write the set builder form of $A = \{-1, 1\}$

(a) $A = \{x : x$ is an integer$\}$
(b) $A = \{x : x$ is a root of the equation $x^2 + 1 = 0\}$
(c) $A = \{x : x$ is a real number$\}$
(d) $A = \{x : x$ is a root of the equation $x^2 = 1\}$

39. If the operation $\oplus$ is defined by $a \oplus b = a^2 + b^2$ for all real numbers 'a' and 'b', then $(2 \oplus 3) \oplus 4 =$ _____

(a) 182 (b) 185 (c) 181 (d) 184

40. If $z = \frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$, then $|z|$ is equal to

(a) 1 (b) 3 (c) 0 (d) 2

41. If α and β are the roots of $x^2 - ax + b^2 = 0$, then $\alpha^2 + \beta^2$ is equal to _____

(a) $2a^2 - b^2$ (b) $a^2 + b^2$ (c) $a^2 - 2b^2$ (d) $a^2 - b^2$

42. If the 2^{nd} and 5^{th} terms of a G.P. are 24 and 3 respectively, then the sum of 1^{st} six terms is _____

(a) $\frac{189}{5}$ (b) $\frac{2}{189}$ (c) $\frac{189}{2}$ (d) $\frac{179}{2}$

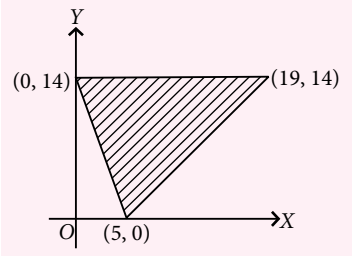
43. The middle term of expansion $\left(\frac{10}{x} + \frac{x}{10}\right)^{10}$ is

(a) ${}^8C_5$ (b) ${}^{10}C_5$ (c) ${}^7C_5$ (d) ${}^9C_5$

44. If $\begin{vmatrix} 2a & x_1 & y_1 \\ 2b & x_2 & y_2 \\ 2c & x_3 & y_3 \end{vmatrix} = \frac{abc}{2} \neq 0$, then the area of the triangle whose vertices are $\left(\frac{x_1}{a}, \frac{y_1}{a}\right)$, $\left(\frac{x_2}{b}, \frac{y_2}{b}\right)$ and $\left(\frac{x_3}{c}, \frac{y_3}{c}\right)$ is

(a) $\frac{1}{8}abc$ (b) $\frac{1}{8}$ (c) $\frac{1}{4}abc$ (d) $\frac{1}{4}$

45. The shaded region shown in fig. is given by the inequation



(a) $14x + 5y \geq 70$, $y \leq 14$ and $x - y \geq 5$
(b) $14x + 5y \geq 70$, $y \geq 14$ and $x - y \geq 5$
(c) $14x + 5y \geq 70$, $y \leq 14$ and $x - y \leq 5$
(d) $14x + 5y \leq 70$, $y \leq 14$ and $x - y \geq 5$

46. $\sim[(\sim p) \wedge q]$ is logically equivalent to

(a) $p \wedge (\sim q)$ (b) $\sim(p \vee q)$
(c) $p \vee (\sim q)$ (d) $\sim[p \wedge (\sim q)]$

47. The value of $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right) + \sin^{-1}\left(\frac{1}{3}\right)$ is equal to

(a) $\frac{\pi}{2}$ (b) $\frac{2\pi}{3}$ (c) $\frac{\pi}{6}$ (d) $\frac{\pi}{4}$

48. If the eccentricity of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{5}{4}$ and $2x + 3y - 6 = 0$ is a focal chord of the hyperbola, then the length of transverse axis is equal to _____.

(a) $\frac{24}{5}$ (b) $\frac{5}{24}$ (c) $\frac{12}{5}$ (d) $\frac{6}{5}$

49. If $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$, $|\vec{b}| = 5$ and the angle between $\vec{a}$ and $\vec{b}$ is $\frac{\pi}{6}$, then the area of the triangle formed by these two vectors as two sides is

(a) 15 (b) $\frac{15\sqrt{3}}{2}$ (c) $\frac{15}{2}$ (d) $\frac{15}{4}$

50. Let $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$. If $\vec{b}$ is a vector such that $\vec{a} \cdot \vec{b} = |\vec{b}|^2$ and $|\vec{a} - \vec{b}| = \sqrt{7}$, then $|\vec{b}| =$ _____

(a) 14 (b) 21 (c) 7 (d) $\sqrt{7}$

51. If direction cosines of a vector of magnitude 3 are $\frac{2}{3}, -\frac{9}{3}, \frac{2}{3}$ and $a > 0$, then vector is _____.

(a) $2\hat{i} - \hat{j} + 2\hat{k}$ (b) $\hat{i} + 2\hat{j} + 2\hat{k}$
(c) $2\hat{i} + \hat{j} + 2\hat{k}$ (d) $\hat{i} - 2\hat{j} + 2\hat{k}$

52. Equation of line passing through the point (2, 3, 1) and parallel to the line of intersection of the plane $x - 2y - z + 5 = 0$ and $x + y + 3z = 6$ is

(a) $\frac{x-2}{-5}=\frac{y-3}{-4}=\frac{z-1}{3}$ (b) $\frac{x-2}{4}=\frac{y-3}{3}=\frac{z-1}{2}$

(c) $\frac{x-2}{5}=\frac{y-3}{-4}=\frac{z-1}{3}$ (d) $\frac{x-2}{5}=\frac{y-3}{4}=\frac{z-1}{3}$

53. Foot of perpendicular drawn from the origin to the plane $2x - 3y + 4z = 29$ is _____.

(a) (2, –3, 4) (b) (5, –2, 3) (c) (5, –1, 4) (d) (7, –1, 3)

54. If two dice are thrown simultaneously, then the probability that the sum of the numbers which come up on the dice to be more than 5 is _____.

(a) $\frac{1}{6}$ (b) $\frac{13}{18}$ (c) $\frac{5}{36}$ (d) $\frac{5}{18}$

55. If $y = f(x^2 + 2)$ and $f'(3) = 5$, then $\frac{dy}{dx}$ at $x = 1$ is _____.

(a) 25 (b) 10 (c) 5 (d) 15

56. If $x = a\cos^3\theta$, $y = a\sin^3\theta$, then $1+\left(\frac{dy}{dx}\right)^2$ is _____.

(a) $\tan^2\theta$ (b) 1 (c) $\tan\theta$ (d) $\sec^2\theta$

57. Slope of normal to the curve $y = x^2 - \frac{1}{x^2}$ at (–1, 0) is

(a) $-\frac{1}{4}$ (b) –4 (c) $\frac{1}{4}$ (d) 4

58. $\int \frac{1}{x^2(x^4+1)^{3/4}}\,dx$ is equal to _____.

(a) $\frac{-(1+x^4)^{1/4}}{x^2}+C$ (b) $\frac{-(1+x^4)^{3/4}}{x}+C$

(c) $\frac{-(1+x^4)^{1/4}}{x}+C$ (d) $\frac{-(1+x^4)^{1/4}}{2x}+C$

59. If $f : R \to R$ is defined by $f(x) = \frac{x}{x^2+1}$, find $f(f(2))$.

(a) $\frac{10}{29}$ (b) 29 (c) $\frac{1}{29}$ (d) $\frac{29}{10}$

60. Evaluate $\begin{vmatrix} \cos 15^\circ & \sin 15^\circ \\ \sin 75^\circ & \cos 75^\circ \end{vmatrix}$

(a) 0 (b) 3 (c) 1 (d) 2

SOLUTIONS

1. (a): Since, the man is one step away from starting point means that either

(i) man has taken 6 steps forward and 5 steps backward

or

(ii) man has taken 5 steps forward and 6 steps backward.

Taking, movement 1 step forward as success and 1 step backward as failure

$\therefore$ p = probability of success = 0.4

q = probability of failure = 0.6

$\therefore$ Required probability

$= p(X = 6 \text{ or } X = 5)$

$= p(X = 6) + p(X = 5)$

$= {}^{11}C_6\, p^6q^5 + {}^{11}C_5\, p^5q^6$

$= {}^{11}C_6\,[p^6q^5 + p^5q^6]$

$= {}^{11}C_6\,[(0.4)^6\,(0.6)^5 + (0.4)^5\,(0.6)^6]$

$= {}^{11}C_6\,(0.4)^5\,(0.6)^5\,[0.4 + 0.6]$

$= {}^{11}C_6\,(0.24)^5$

2. (d): Let $I = \int_0^{\pi/4} \log\left(\frac{\sin x+\cos x}{\cos x}\right)dx$

$= \int_0^{\pi/4} \log(1+\tan x)\,dx$

Using property, $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$\Rightarrow I = \int_0^{\pi/4} \log\left(1+\tan\left(\frac{\pi}{4}-x\right)\right)dx$

$= \int_0^{\pi/4} \log\left(1+\frac{1-\tan x}{1+\tan x}\right)dx$

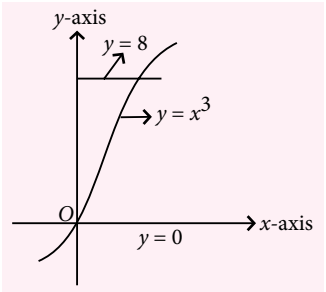
$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan x}\right)dx$

$= \int_0^{\pi/4} \log 2\,dx - \int_0^{\pi/4} \log(1+\tan x)\,dx$

$\Rightarrow I = \log 2\cdot\frac{\pi}{4} - I \Rightarrow 2I = \frac{\pi}{4}\log 2 \Rightarrow I = \frac{\pi}{8}\log 2$

3. (d): Given equation of curve is

$y = x^3 \Rightarrow x = (y)^{1/3}$



$\therefore$ Required bounded area

$= \int_0^8 x\,dy = \int_0^8 y^{1/3}\,dy = \frac{3}{4}\left|y^{4/3}\right|_0^8$

$= \frac{3}{4}\left[(8)^{4/3}\right] = \frac{3}{4}\times 16 = 12$ sq. units

4. (b): $\vec{a}=\hat{i}+2\hat{j}+\hat{k}$, $\vec{b}=\hat{i}-\hat{j}+\hat{k}$

$\vec{c}=\hat{i}+\hat{j}-\hat{k}$

Any vector in the plane of $\vec{a}$ and $\vec{b}$ is

$\vec{r}=\vec{a}+\lambda\vec{b}$

$=(1+\lambda)\hat{i}+(2-\lambda)\hat{j}+(1+\lambda)\hat{k}$

$\therefore$ Projection of $\vec{r}$ on $\vec{c}$ $=\left|\frac{(\vec{r}\cdot\vec{c})}{|\vec{c}|}\right|=\frac{1}{\sqrt{3}}$

$\Rightarrow \left|\frac{1(1+\lambda)+1(2-\lambda)-1(1+\lambda)}{\sqrt{1^2+1^2+(-1)^2}}\right|=\frac{1}{\sqrt{3}}$

$\Rightarrow \left|\frac{2-\lambda}{\sqrt{3}}\right|=\frac{1}{\sqrt{3}}$

$\Rightarrow 2-\lambda=\pm 1$

$\Rightarrow \lambda = 1$ or 3

Hence, $\vec{r}=2\hat{i}+\hat{j}+2\hat{k}$ or $4\hat{i}-\hat{j}+4\hat{k}$

5. (b): Given data is 3, 10, 10, 4, 7, 10, 5

$n = 7$

Mean $= \bar{x}=\frac{3+10+10+4+7+10+5}{7}$

$=\frac{49}{7}=7$

Mean deviation from mean $=\frac{\sum|x_i-\bar{x}|}{n}$

$=\frac{|3-7|+|10-7|+|10-7|+|4-7|+|7-7|+|10-7|+|5-7|}{7}$

$=\frac{4+3+3+3+3+2}{7}=\frac{18}{7}=2.57$

6. (c): Since, sum of all probabilities

$=\Sigma P(x) = 1$

$\Rightarrow 0.2 + k + k + 2k = 1$

$\Rightarrow 4k = 0.8 \Rightarrow k=\frac{0.8}{4}=0.2$

7. (c): Given, $g(x)=\frac{x^{200}}{200}+\frac{x^{199}}{199}+.....+\frac{x^2}{2}+x+5$

$\Rightarrow g'(x)=\frac{200}{200}x^{199}+\frac{199}{199}x^{198}+.....+1$

$\Rightarrow g'(x) = x^{199} + x^{198} + ... + 1$

$\Rightarrow g'(0) = 1$

8. (a): Required Probability $=\frac{{}^2C_1}{{}^{10}C_1}=\frac{2}{10}=\frac{1}{5}$

9. (d):

10. (b): Since $f(x)$ is continuous

$\therefore \lim\limits_{x\to 5^-} f(x)=\lim\limits_{x\to 5^+} f(x)$

$\Rightarrow \lim\limits_{x\to 5^-} 3x-8=2k$

$\Rightarrow 3(5) - 8 = 2k$

$\Rightarrow 2k = 7 \Rightarrow k=\frac{7}{2}$

11. (d): Given, $f(x) = 2x^2$

$\Rightarrow f(3.8) = 2 \times 3.8 \times 3.8 = 28.88$

$\Rightarrow f(4) = 2 \times 4 \times 4 = 32$

$\therefore \frac{f(3.8)-f(4)}{3.8-4}=\frac{28.88-32}{3.8-4}=\frac{3.12}{0.2}=15.6$

12. (d): Given, $x = ct$ and $y=\frac{c}{t}$

$\Rightarrow \frac{dx}{dt}=c$ ...(i) and $\frac{dy}{dt}=\frac{-c}{t^2}$ ...(ii)

Dividing (ii) by (i), we get

$\frac{dy}{dx}=\frac{-1}{t^2} \Rightarrow \left(\frac{dy}{dx}\right)_{t=2}=\frac{-1}{4}$

13. (c): According to question,

$\frac{dV}{dt}=10$, where V is volume of spherical balloon

$\Rightarrow \frac{d\left(\frac{4}{3}\pi r^3\right)}{dt}=10 \Rightarrow \frac{4}{3}\pi\times 3r^2\frac{dr}{dt}=10$

$\Rightarrow \frac{dr}{dt}=\frac{5\times 3}{2\pi\times 3r^2}=\frac{5}{2\pi r^2}$

At $r = 15$ cm

$\frac{dr}{dt}=\frac{5}{2\pi\times 15\times 15}=\frac{1}{90\pi}$ cm/sec

14. (a): Let $I=\int\frac{\sin^2 x}{1+\cos x}\,dx$

$=\int\frac{1-\cos^2 x}{1+\cos x}\,dx$ $(\because \sin^2x + \cos^2x = 1)$

$=\int(1 - \cos x)dx = x - \sin x + C$

15. (c) **16. (b)**

17. (b): Given, trigonometrical equation is

$\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$

Put $x = \tan\theta$, then

$\tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right)=\frac{1}{2}\tan^{-1}(\tan\theta)$

$\Rightarrow \tan^{-1}\left(\tan\left(\frac{\pi}{4}-\theta\right)\right)=\frac{1}{2}\tan^{-1}(\tan\theta)$

$\Rightarrow \frac{\pi}{4}-\theta=\frac{1}{2}\theta \Rightarrow \frac{3}{2}\theta=\frac{\pi}{4} \Rightarrow \theta=\frac{\pi}{6}$

$\Rightarrow \tan^{-1}x=\frac{\pi}{6} \Rightarrow x=\tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$

18. (a) : Given system of equations are

$$\left.\begin{aligned} x+y+z&=6\\ x+2y+3z&=10\\ x+2y+az&=b \end{aligned}\right\} \quad ...(i)$$

(i) will have no solution when $D = 0$ and atleast one of D_1, D_2 or D_3 is non-zero.

Now, $D=\begin{vmatrix}1&1&1\\1&2&3\\1&2&a\end{vmatrix}=0$

$\Rightarrow$ $1(2a - 6) - 1(a - 3) + 1(2 - 2) = 0$

$\Rightarrow$ $2a - 6 - a + 3 = 0$

$\Rightarrow$ $a - 3 = 0 \Rightarrow a = 3$ ...(ii)

Let $D_1 \neq 0$, then $\begin{vmatrix}6&1&1\\10&2&3\\b&2&a\end{vmatrix}\neq 0$

$\Rightarrow$ $6(2a - 6) - 10(a - 2) + b(3 - 2) \neq 0$

$\Rightarrow$ $2a + b - 16 \neq 0 \Rightarrow b - 10 \neq 0$ [from (ii)]

$\Rightarrow$ $b \neq 10$

19. (a)

20. (c) : We have, $\frac{(x+1)^2}{x^3+x}=\frac{A}{x}+\frac{Bx+C}{x^2+1}$

$\Rightarrow$ $(x + 1)^2 = A(x^2 + 1) + (Bx + C)x$

$\Rightarrow$ $x^2 + 2x + 1 = (A + B)x^2 + Cx + A$ ...(i)

Comparing coefficients of x^2, x and constant, we get

$A + B = 1$, $C = 2$, $A = 1$

Put $A = 1$ in (i) we get $B = 0$

Now, $\text{cosec}^{-1}\left(\frac{1}{A}\right)+\cot^{-1}\left(\frac{1}{B}\right)+\sec^{-1}C$

$= \text{cosec}^{-1}(1) + \cot^{-1}(\infty) + \sec^{-1}(2)$

$=\frac{\pi}{2}+0+\frac{\pi}{3}=\frac{5\pi}{6}$

21. (c) : $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24 = 12 \times 2$,

$5! = 120 = 12 \times 10$, $6! = 720 = 12 \times 60$, ...

$\therefore$ $1! + 2! + 3! + ... + 11!$

$= 1 + 2 + 6 + (12 \times 2) + (12 \times 10) + (12 \times 60) + ... + (12 \times p)$ $[11! = 12p]$

$= (1 + 2 + 6) + 12[2 + 10 + 60 + ... + p]$

$= 9 + 12q$, where $p \in N$ and $q \in N$

$\therefore$ when $1! + 2! + 3! + ... + 11!$ is divided by 12 then remainder is 9

22. (d): Given, $\alpha \leq 2\sin^{-1}x + \cos^{-1}x \leq \beta$

Since $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}$

$\therefore$ $\alpha \leq \frac{\pi}{2} + \sin^{-1}x \leq \beta$ ...(i)

As $\frac{-\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$ ...(ii)

Adding $\frac{\pi}{2}$ in (ii), we get

$0 \leq \frac{\pi}{2} + \sin^{-1}x \leq \pi$ ...(iii)

By comparing (i) and (iii), we get $\alpha = 0$, $\beta = \pi$

23. (d): We have $A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$

So, $A^2=\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}$

24. (b): $f(x) = [x]$ is continuous at $x = 1.5$

$\because$ $\lim\limits_{x\to 1.5^-}[x]=1$ and $\lim\limits_{x\to 1.5^+}[x]=1$

25. (c)

26. (a) : For given equation $x^3 - 3xy^2 + 2 = 0$

Taking derivative w.r.t. 'x' on both sides, we get

$$3x^2-3\left[x\cdot 2y\frac{dy}{dx}+y^2\right]=0$$

$\Rightarrow$ $x^2-2xy\frac{dy}{dx}-y^2=0$

$\Rightarrow$ $\frac{dy}{dx}=\frac{x^2-y^2}{2xy}=m_1$ (say)

For equation $3x^2y - y^3 = 2$

Taking derivative w.r.t. 'x' on both sides

$\Rightarrow$ $3x^2\frac{dy}{dx}+6xy-3y^2\frac{dy}{dx}=0$

$\Rightarrow$ $\frac{dy}{dx}(x^2-y^2)=-2xy$

$\Rightarrow$ $\frac{dy}{dx}=\frac{-2xy}{(x^2-y^2)}=m_2$ (say)

Since $m_1 \times m_2 = -1$

$\therefore$ Both the curves cut at right angle.

27. (c) : Let $y = x^2 - 8x + 17 \Rightarrow \frac{dy}{dx}=2x-8$

Put $\frac{dy}{dx}=0 \Rightarrow 2x - 8 = 0 \Rightarrow x = 4$

Since $\frac{d^2y}{dx^2} = 2 > 0$

$\therefore$ y is minimum at $x = 4$

$\therefore$ $(\text{Min } y)_{x=4} = (4)^2 - 8(4) + 17 = 16 - 32 + 17 = 1$

28. (a) **29. (a)**

30. (b): Given d.e. is

$$x\frac{dy}{dx}+2y=x^2 \quad \text{or} \quad \frac{dy}{dx}+\frac{2}{x}y=x$$

$$\text{I.F.}=e^{\int\frac{2}{x}dx}=e^{2\log x}=x^2$$

$\therefore$ Required solution is $y\cdot x^2=\int x\cdot x^2dx+k$

or $yx^2=\frac{x^4}{4}+k$ or $y=\frac{x^4+C}{4x^2}$

31. (b) **32. (d)** **33. (b)**

34. (d): Given equation of line is $3x + y = 3$

Its slope $= -3$

Slope of perpendicular line $= \frac{1}{3}$

$\therefore$ Equation of line passing through (2, 2) having slope $\frac{1}{3}$ is $y - 2 = \frac{1}{3}(x - 2)$

or, $3y - 6 = x - 2$ or, $x - 3y = -4$

or, $\frac{x}{-4}+\frac{y}{\left(\frac{4}{3}\right)}=1$

Compare with intercept form of line $\frac{x}{a}+\frac{y}{b}=1$,

we get y-intercept $= \frac{4}{3}$

35. (b): Given function is $f(x)=\frac{1}{x}$

Since $f: R \to R$, but for $x = 0$, function is not defined

Also, we do not get any image for $x = 0$

$\therefore$ f is not defined in the given interval.

36. (d): Given inequation is $\frac{x^2+6x-7}{|x+4|}<0$ ...(i)

Since $|x + 4| > 0\ \forall x \in R$ except for $x = -4$

$\because$ for $x = -4$, (i) do not exist.

$\therefore$ $x^2 + 6x - 7 < 0$ for given inequation

or $x^2 + 7x - x - 7 < 0$

or $(x + 7)(x - 1) < 0$

or $x \in (-7, 1)$

$\therefore$ Solution set of (i) is $(-7, -4) \cup (-4, 1)$

37. (c): Given functions are

$$f(x)=\frac{1}{2}-\tan\left(\frac{\pi x}{2}\right),\ -1<x<1$$

Since $f(x)$ is defined in the given interval $-1 < x < 1$

$\therefore$ its domain $= (-1, 1)$ ...(i)

$$g(x)=\sqrt{3+4x-4x^2}$$

Since $g(x)$ is defined when

$3 + 4x - 4x^2 \geq 0$

$\Rightarrow$ $4x^2 - 4x - 3 \leq 0$ $\Rightarrow$ $4x^2 - 6x + 2x - 3 \leq 0$

$\Rightarrow$ $(2x - 3)(2x + 1) \leq 0$

$\Rightarrow$ $\frac{-1}{2}\leq x\leq\frac{3}{2}$ ...(ii)

From (i) and (ii), we get

Domain of $(f + g) = \left[-\frac{1}{2}, 1\right)$

38. (d): Since $x = -1$ and $x = 1$ are roots

$\therefore$ $(x + 1)$ and $(x - 1)$ are factors of equation.

$\therefore$ $(x + 1)(x - 1) = x^2 - 1 = 0$ is required equation.

$\therefore$ Set builder form of given set A is

$A = \{x : x$ is a root of the equation $x^2 = 1\}$

39. (b)

40. (d): $z=\frac{(\sqrt{3}+i)^3(3i+4)^2}{(8+6i)^2}$

Taking Modulus on both sides, we get

$$|z|=\frac{|\sqrt{3}+i|^3|3i+4|^2}{|8+6i|^2}=\frac{(\sqrt{3+1})^3(\sqrt{9+16})^2}{(\sqrt{64+36})^2}=\frac{8\times25}{100}=2$$

41. (c): Given equation is

$x^2 - ax + b^2 = 0$

Since α and β are its roots

$\therefore$ $\alpha + \beta = a$, $\alpha\beta = b^2$

Now, $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta = a^2 - 2b^2$

42. (c): Let 2^{nd} term $= a_2 = 24$, 5^{th} term $= a_5 = 3$

$\Rightarrow$ $ar = 24$ ...(i)

and $ar^4 = 3$ ...(ii)

where a and r are first term and c.r. of G.P.

Dividing (ii) by (i), we get $r^3=\frac{3}{24}=\frac{1}{8}\Rightarrow r=\frac{1}{2}$

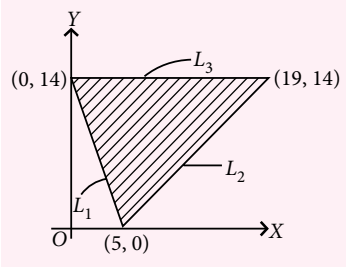
From (i), $a\cdot\frac{1}{2}=24$ or $a = 48$

$$\therefore\ S_6=\frac{a(1-r^6)}{1-r}=\frac{48\left(1-\left(\frac{1}{2}\right)^6\right)}{\left(1-\frac{1}{2}\right)}$$

$$=(48\times2)\left(1-\frac{1}{64}\right)=48\times2\times\frac{63}{64}=\frac{189}{2}$$

43. (b) **44. (b)**

45. (c):



Line (L_1) represent an equation $14x + 5y = 70$
Since shaded region (L_1) does not contain origin
$\therefore$ its corresponding inequation is
$14x + 5y \geq 70$ ...(i)
Line (L_2) represent an equation $x - y = 5$
Since shaded region (L_2) contain the origin
$\therefore$ Its corresponding inequation is $x - y \leq 5$ ...(ii)
Line (L_3) represent $y = 14$. Its shaded region contain the origin
$\therefore$ Its corresponding inequation is $y \leq 14$ ...(iii)
$\therefore$ From (i), (ii) and (iii), we get
$14x + 5y \geq 70$, $x - y \leq 5$ and $y \leq 14$

46. (c) : $\sim[(\sim p) \wedge q] = \sim(\sim p) \vee (\sim q) = p \vee (\sim q)$

47. (a) : $\sin^{-1}\left(\frac{2\sqrt{2}}{3}\right)+\sin^{-1}\left(\frac{1}{3}\right)$

$= \sin^{-1}\left(\left(\frac{2\sqrt{2}}{3}\right)\sqrt{1-\left(\frac{1}{3}\right)^2}+\left(\frac{1}{3}\right)\sqrt{1-\left(\frac{2\sqrt{2}}{3}\right)^2}\right)$

$= \sin^{-1}\left(\left(\frac{2\sqrt{2}}{3}\right)\sqrt{1-\frac{1}{9}}+\left(\frac{1}{3}\right)\sqrt{1-\frac{8}{9}}\right)$

$= \sin^{-1}\left(\frac{2\sqrt{2}}{3}\times\frac{2\sqrt{2}}{3}+\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\right)$

$= \sin^{-1}\left(\frac{8}{9}+\frac{1}{9}\right)=\sin^{-1}(1)=\frac{\pi}{2}$

48. (a) : Since co-ordinates of focus $(ae, 0)$

$= \left(\frac{5a}{4}, 0\right)$

If $2x + 3y - 6 = 0$ ...(i)
is a focal chord of hyperbola, then
$\left(\frac{5a}{4}, 0\right)$ will pass through equation (i)

So, $2\times\frac{5a}{4}+0-6=0$

$\Rightarrow \frac{5a}{2}-6=0 \Rightarrow a=\frac{2\times6}{5}=\frac{12}{5}$

$\therefore$ Length of transverse axis $= 2a=2\times\frac{12}{5}=\frac{24}{5}$

49. (d)

50. (d) : We have, $|\vec{a}-\vec{b}|=\sqrt{7}$
Squaring both sides, we get

$|\vec{a}-\vec{b}|^2=\left(\sqrt{7}\right)^2 \Rightarrow |\vec{a}|^2+|\vec{b}|^2-2\vec{a}\cdot\vec{b}=7$

$\Rightarrow (1+4+9)+|\vec{b}|^2-2|\vec{b}|^2=7 \quad (\because \vec{a}\cdot\vec{b} = |\vec{b}|^2)$

$\Rightarrow 14-|\vec{b}|^2=7 \Rightarrow |\vec{b}|^2=7 \Rightarrow |\vec{b}|=\sqrt{7}$

51. (None of the options is correct) : In given question, direction cosines should be $\frac{2}{3}, \frac{-1}{3}, \frac{2}{3}$

($\because$ Sum of squares of d.c's is equal to 1)

$\therefore$ Required vector $= 3\left(\frac{2}{3}\hat{i}-\frac{1}{3}\hat{j}+\frac{2}{3}\hat{k}\right)$

$= 2\hat{i}-\hat{j}+2\hat{k}$

52. (a)

53. (a) : If foot of perpendicular from a point (x_1, y_1, z_1) on the plane $ax + by + cz + d$ is (α, β, γ) then

$$\frac{\alpha-x_1}{a}=\frac{\beta-y_1}{b}=\frac{\gamma-z_1}{c}=\frac{-(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

Let foot of perpendicular be (α, β, γ), then by using above formula, we get

$$\frac{\alpha-0}{2}=\frac{\beta-0}{-3}=\frac{\gamma-0}{4}=\frac{-(2(0)-3(0)+4(0)-29)}{(2)^2+(-3)^2+(4)^2}$$

or $\frac{\alpha}{2}=\frac{\beta}{-3}=\frac{\gamma}{4}=\frac{29}{29}$

$\Rightarrow \alpha = 2, \beta = -3, \gamma = 4$

$\therefore$ Required point is $(2, -3, 4)$

54. (b)

55. (b) : $y = f(x^2 + 2)$

$\Rightarrow \frac{dy}{dx}=f'(x^2+2)\times2x$

$\Rightarrow \left(\frac{dy}{dx}\right)_{\text{at } x=1}=f'(3)\times2=5\times2=10$

56. (d) **57. (c)**

58. (c) : Put $1+\frac{1}{x^4}=t \Rightarrow \frac{-4}{x^5}dx=dt$

$\Rightarrow I=-\frac{1}{4}\int\frac{dt}{(t)^{3/4}}=\frac{-1}{4}\left(\frac{t^{\left(\frac{-3}{4}+1\right)}}{-\frac{3}{4}+1}\right)+C$

$= -t^{1/4}+C=-\left(1+\frac{1}{x^4}\right)^{1/4}+C=-\frac{(x^4+1)^{1/4}}{x}+C$

59. (a) : Given function is $f(x)=\frac{x}{x^2+1} \Rightarrow f(2)=\frac{2}{5}$

Now, $f(f(2))=f\left(\frac{2}{5}\right)=\frac{\left(\frac{2}{5}\right)}{\left(\frac{2}{5}\right)^2+1}=\frac{2}{5}\times\frac{25}{29}=\frac{10}{29}$

60. (a)

■■

ACE YOUR WAY CBSE XI

Series 2

RELATIONS & FUNCTIONS

ORDERED PAIR

A pair of objects written in a particular order is called an ordered pair. Let A & B be two non-empty sets. If $a \in A$ & $b \in B$, then an element of the form (a, b) is called an ordered pair.

CARTESIAN PRODUCT OF SETS

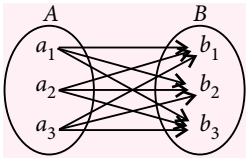
Cartesian product of	Definition	Representation	Example
Two sets	The cartesian product of two non-empty sets A & B is defined as the set of all ordered pairs (a, b), where $a \in A$ & $b \in B$	$A \times B$ $= \{(a, b) : a \in A, b \in B\}$ Thus, $(a, b) \in A \times B$ $\Leftrightarrow a \in A$ & $b \in B$	Let $A = \{1, 2, 3\}$, $B = \{2, 4\}$ then $A \times B = \{1, 2, 3\} \times \{2, 4\}$ $= \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$
Three sets	The cartesian product of three non-empty sets A, B & C is defined as the set of all ordered triplets (a, b, c), where $a \in A$, $b \in B$ & $c \in C$	$A \times B \times C$ $= \{(a, b, c): a \in A, b \in B$ & $c \in C\}$ Thus, $(a, b, c) \in A \times B \times C$ $\Leftrightarrow a \in A, b \in B$ & $c \in C$	Let $A = \{1, 2\}$, $B = \{2, 3\}$ & $C = \{3, 4\}$ $A \times B \times C = A \times (B \times C)$ $= \{1, 2\} \times (\{2, 3\} \times \{3, 4\})$ $= \{1, 2\} \times (\{(2, 3), (2, 4), (3, 3), (3, 4)\})$ $= \{(1, 2, 3), (1, 2, 4), (1, 3, 3),(1, 3, 4),$ $(2, 2, 3), (2, 2, 4), (2, 3, 3), (2, 3, 4)\}$
***n* sets**	The cartesian product of n non-empty sets A_1, A_2,.., A_n is defined as the set of elements $(a_1, a_2,..., a_n)$, where $a_1 \in A_1$, $a_2 \in A_2,..., a_n \in A_n$	$A_1 \times A_2 \times ... \times A_n$ $= \{(a_1, a_2,..., a_n) : a_1 \in A_1,$ $a_2 \in A_2,... a_n \in A_n\}$ or$\{(a_1, a_2, ..., a_n): a_i \in A_i$ for $i = 1, 2, ... n\}$ Thus, $(a_1, a_2, ..., a_n) \in A_1 \times A_2 \times ...$ $... \times A_n$ $\Leftrightarrow a_i \in A_i$ for $i = 1, 2,... ,n$	

Note :

- For two ordered pairs (a_1, b_1) and (a_2, b_2), $(a_1, b_1) = (a_2, b_2) \Leftrightarrow a_1 = a_2, b_1 = b_2$.
- If $A = \phi$ or $B = \phi$, then $A \times B = \phi$.
- $A \times B \neq \phi \Leftrightarrow A \neq \phi$ & $B \neq \phi$.
- If atleast one of A & B is an infinite set and other is a non-empty set, then $A \times B$ is an infinite set.

REPRESENTATION OF CARTESIAN PRODUCT OF TWO SETS BY ARROW DIAGRAM

Let $A = \{a_1, a_2, a_3\}$ & $B = \{b_1, b_2, b_3\}$, then cartesian product of A & B, *i.e.*, $A \times B$ can be represented by arrow diagram as shown in the figure.



In this figure, lines have been drawn from each element of set A to each element of set B.

RELATIONS

A relation R, from a non-empty set A to another non-empty set B, is a subset of $A \times B$.

i.e., Any subset of $A \times B$ is a relation from A to B.

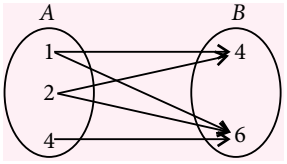
Thus, R is a relation from A to $B \Leftrightarrow R \subseteq A \times B$

$\Leftrightarrow \quad R \subseteq \{(a, b) : a \in A, b \in B\}$

For example,

Let $A = \{1, 2, 4\}$ and $B = \{4, 6\}$

There exists a relation "is less than" from A to B and is represented as $R = \{(1, 4), (1, 6), (2, 4), (2, 6), (4, 6)\}$



DOMAIN, RANGE AND CO-DOMAIN OF A RELATION

	Definition	Representation	Examples
Domain	For any relation R from A to B, the domain is the set of all those elements $a \in A$ s.t. $(a, b) \in R$ for some $b \in B$.	Domain of R $= \{a \in A : (a, b) \in R$ for some $b \in B\}$	Let $A = \{2, 3, 5\}$, $B = \{4, 7, 8, 10\}$ Let $a\ R\ b \Leftrightarrow a$ divides b then $R = \{(2, 4), (2, 8), (2, 10), (5, 10)\}$ Here domain of $R = \{2, 5\}$
Range	For any relation R from A to B, the range is the set of all those elements $b \in B$ s.t. $(a, b) \in R$ for some $a \in A$.	Range of R $= \{b \in B : (a, b) \in R$ for some $a \in A\}$	In above example, Range of $R = \{4, 8, 10\}$
Co-domain	For any relation R from A to B, B is called the co-domain of R.	Co-domain of R $= \{b \ \forall\ b \in B\}$ $= B$	In above example, Co-domain of $R = B = \{4, 7, 8, 10\}$

Remarks :

- A relation may be represented algebraically either by the Roaster method or by the set-builder method.
- An arrow diagram is a visual representation of a relation.
- Range of $R \subseteq$ co-domain of R

Number of Relations

Let A & B be two non-empty sets having elements p & q respectively.

Then $n(A \times B) = n(A) \times n(B) = pq$

$\therefore$ Total number of subsets of $A \times B = 2^{pq}$

Since, each subset of $A \times B$ is a relation from A to B

$\therefore$ Total number of relations from A to $B = 2^{pq}$.

FUNCTIONS

- A relation f from a set A to a set B is said to be a function if every element of set A has one and only one image in set B.
- A function f is a relation from a non-emtpy set A to a non-empty set B such that the domain of f is A and no two distinct ordered pair in f have the same first element.

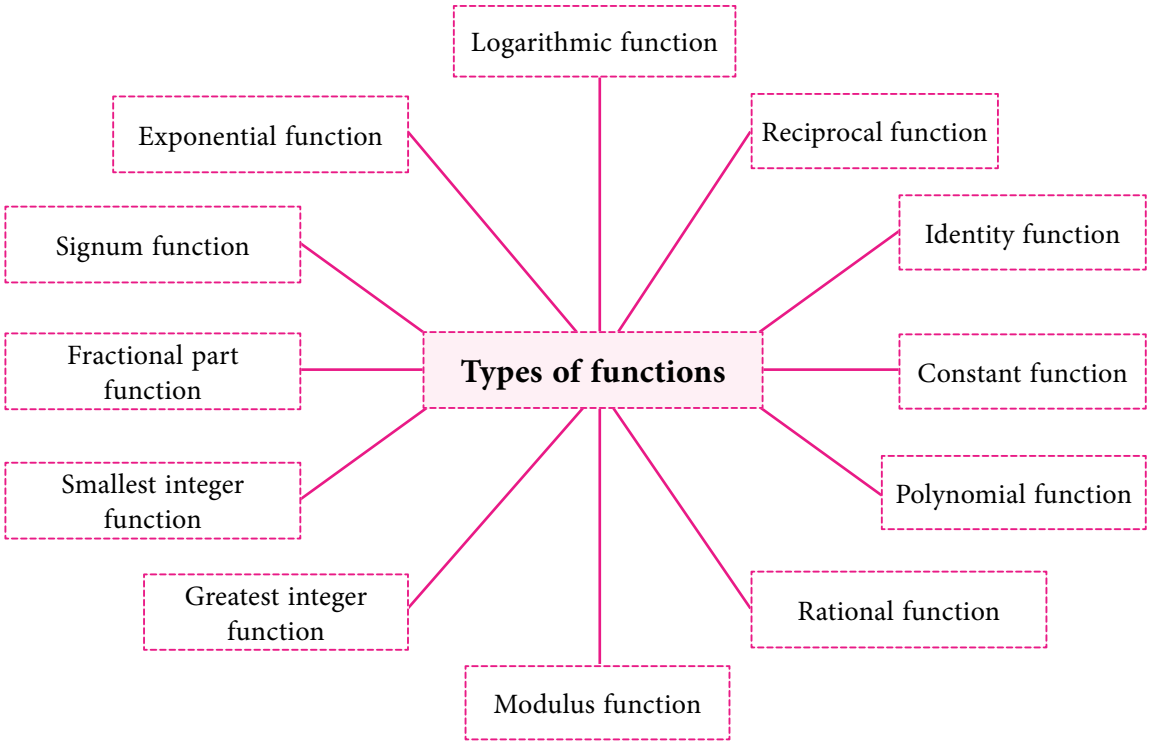
$f : A \to B$ s.t. $x \in A$ & $y \in B$, where $y = f(x)$

DOMAIN, RANGE AND CO-DOMAIN OF A FUNCTION

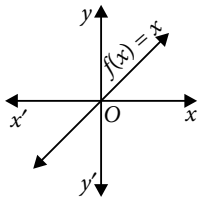
Domain	The set A is called the domain of the function f.
Co-domain	The set B is called the co-domain of the function f.
Range	The set of the images of all the elements of A under the map f is called the range of function f. Range of f *i.e.*, $f(A) = \{f(x) : x \in A\}$. Also $f(A) \subseteq B$

REAL VALUED FUNCTIONS

- A function which has either R or one of its subsets as its range is called a real valued function. Further, if its domain is also either R or a subset of R, it is called a real valued function.

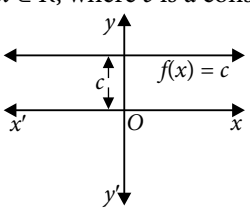


1. **Identity Function :** The function $f : R \to R$ defined by $y = f(x) = x$ for each $x \in R$.



Domain of $f = R$, Range of $f = R$.

2. **Constant Function :** The function $f : R \to R$ defined by $y = f(x) = c$, $x \in R$, where c is a constant $\in R$.



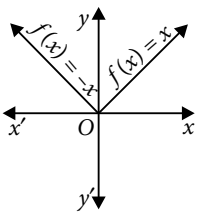
Domain of $f = R$, Range of $f = \{c\}$.

3. **Polynomial Function :** A real valued function $f : R \to R$ defined by $y = f(x) = a_0 + a_1x + ... + a_nx^n$, where $n \in N$ and $a_0, a_1, a_2, ..., a_n \in R$ for each $x \in R$.

4. **Rational Function :** These are the real functions of the type $\frac{f(x)}{g(x)}$, where $f(x)$ and $g(x)$ are polynomial functions of x defined in a domain, where $g(x) \neq 0$.

5. **Modulus Function :** The real function $f : R \to R$ defined by

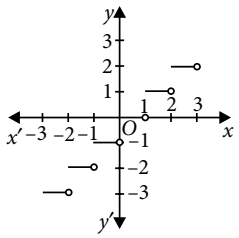
$$f(x) = |x| = \begin{cases} x, x \geq 0 \\ -x, x < 0 \end{cases} \forall x \in R.$$



Domain of $f = R$.

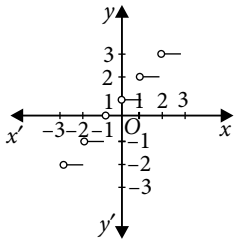
Range of $f = R^+ \cup \{0\}$.

6. **Greatest Integer Function (Floor Function) :** The real function $f : R \to R$ defined by $f(x) = [x]$, $x \in R$ assumes the value of the greatest integer, less than or equal to x.



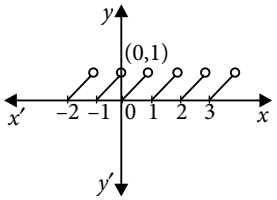
Domain of $f = R$, Range of $f = Z$.

7. **Smallest Integer Function (Ceiling or Step Function) :** The real function $f : R \to R$ defined by $f(x) = \lceil x \rceil \forall\ x \in R$, assumes the value of the least integer, greater than or equal to x.



Domain of $f = R$, Range of $f = Z$.

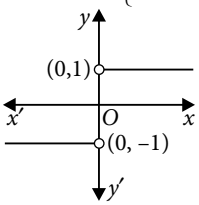
8. **Fractional Part Function :** The function $f : R \to R$ defined by $f(x) = \{x\} \forall\ x \in R$.



Domain of $f = R$, Range of $f = [0, 1)$.

9. **Signum Function :** The real function $f : R \to R$ defined by

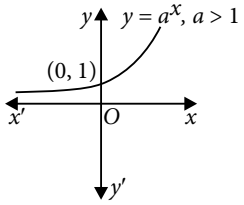
$$f(x) = \begin{cases} \dfrac{|x|}{x}, x \neq 0 \\ 0, \ x = 0 \end{cases} = \begin{cases} 1, \text{ if } x > 0 \\ 0, \text{if } x = 0 \\ -1, \text{if } x < 0 \end{cases}$$



Domain of $f = R$, Range of $f = \{-1, 0, 1\}$.

10. **Exponential Function :** A real function $f : R \to R$ defined by $f(x) = a^x$, where $a > 0$ and $a \neq 1$
Domain of $f = R$
Range of $f = (0, \infty)$

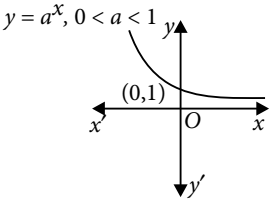
Case I : When $a > 1$



We observe that the values of $y = f(x) = a^x$ increase as the values of x increase.

Also, $f(x) = a^x \begin{cases} < 1, \text{ for } x < 0 \\ = 1, \text{ for } x = 0 \\ > 1, \text{ for } x > 0. \end{cases}$

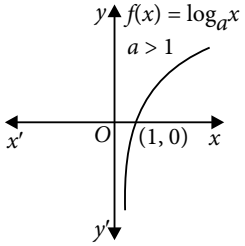
Case II : When $0 < a < 1$



In this case, the values of $y = f(x) = a^x$ decrease with the increase in x and $y > 0\ \forall\ x \in R$.

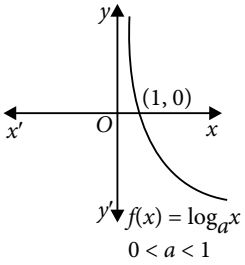
Also, $y = f(x) = a^x \begin{cases} > 1, \text{ for } x < 0 \\ = 1, \text{ for } x = 0 \\ < 1, \text{ for } x > 0 \end{cases}$

11. **Logarithmic Function :** If $a > 0$ and $a \neq 1$, then the function defined by $f(x) = \log_a x, x > 0$.
Domain of $f = (0, \infty)$. Range of $f = R$.
Case I : When $a > 1$



$$y = f(x) = \log_a x \begin{cases} < 0, \text{for } 0 < x < 1 \\ = 0, \text{for } x = 1 \\ > 0, \text{for } x > 1 \end{cases}$$

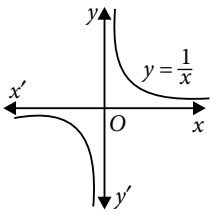
Case II : When $0 < a < 1$



$$y = f(x) = \log_a x \begin{cases} > 0, \text{for } 0 < x < 1 \\ = 0, \text{for } x = 1 \\ < 0, \text{for } x > 1 \end{cases}$$

12. Reciprocal Function : The real function

$f : R - \{0\} \to R$ defined by $f(x) = \frac{1}{x}$



Domain of $f = R - \{0\}$, Range of $f = R - \{0\}$.

ALGEBRA OF REAL FUNCTIONS

Addition of two real functions	Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$, then $(f + g) : X \to R$ defined as $(f + g)(x) = f(x) + g(x)$, for all $x \in X$.
Subtraction of a real function from another	Let $f : X \to R$ and $g : X \to R$ be any two real functions, where $X \subset R$, then $(f - g) : X \to R$ defined as $(f - g)(x) = f(x) - g(x)$, for all $x \in X$.
Multiplication by a scalar	Let $f : X \to R$ be a real valued function and α be a scalar, then $\alpha \cdot f : X \to R$ defined as $(\alpha f)(x) = \alpha f(x)$, for all $x \in X$.
Multiplication of two real functions	Let $f : X \to R$ and $g : X \to R$ be two real functions then $fg : X \to R$ defined as $(fg)(x) = f(x)\, g(x)$, for all $x \in X$.
Quotient of two real functions	Let $f : X \to R$ and $g : X \to R$ be two real functions, then $\frac{f}{g} : X \to R$ defined as $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, provided $g(x) \neq 0$, $x \in X$.

Very Short Answer Type

1. If $(2a - 3, b + 2) = (3a + 1, 2b + 5)$, then find a and b.

2. If $U = \{1, 2, 3, 4\}$ and $R = \{(x, y) : y > x \text{ for all } x, y \in U\}$, then find the domain and range of R.

3. Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Find the number of relations from A to B.

4. Let $A = \{1, 2, 3\}$ and $B = \{4, 5\}$. Find $A \times B$, $B \times A$ and show that $A \times B \neq B \times A$.

5. Which of the following are functions if $X = \{a, b, c, d\}$, $Y = \{1, 2, 3, 4, 5\}$.
(i) $f_1 = \{(a, 1), (b, 1), (c, 3), (d, 4)\}$
(ii) $f_2 = \{(a, 1,), (b, 2), (c, 4), (a, 2), (d, 5)\}$.

Short Answer Type

6. Let R be a relation from $A = \{1, 2, 3\}$ to $B = \{1, 2, 5, 6\}$. Write R as a set of ordered pairs when, (i) R is the relation 'is less than', (ii) R is the relation 'is greater than'. Also find R^{-1} in each case.

7. Find the domain and range of $f(x) = \sqrt{1 - x^2}$ from R to R.

8. (i) If $f(x) = \frac{x^2 - x + 1}{x^2 + x + 1}$, find $f(1 + h)$.
(ii) If $f(x) = \frac{\sin x - \cos x}{\sin x + \cos x}$, find $f\left(\frac{\pi}{3}\right)$.
(iii) If $f(1 + x) = x^2 + 1$, find $f(2 - x)$.
(iv) If $f(x) = \frac{x + 1}{x - 1}$, find $f(x^2)$, $[f(x)]^2$.

9. Let $f = \left\{\left(x, \frac{x^2}{1 + x^2}\right) : x \in R\right\}$ be a function from R into R. Determine the domain and range of f.

10. What are the sum and difference of the identity function and the reciprocal function?

Long Answer Type

11. Find the domain of definition of the function $f(x)$ given by $f(x) = \frac{1}{\log_{10}(1 - x)} + \sqrt{x + 2}$

12. Find the range of each of the following functions:
(i) $f(x) = |x - 3|$
(ii) $f(x) = 1 - |x - 2|$
(iii) $f(x) = \frac{|x - 4|}{x - 4}$

13. Let R be the relation on the set Z of all integers defined by $(x, y) \in R \Rightarrow x - y$ is divisible by n.
Prove that :
(i) $(x, x) \in R$ for all $x \in Z$
(ii) $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$
(iii) $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$ for all $x, y, z \in R$.

14. Find $f + g$, cf $(c \in R, c \neq 0)$ and $\frac{f}{g}$ in each of the following :
(i) If $f(x) = x^3 + 1$ and $g(x) = x + 1$

(ii) If $f(x) = \sqrt{x-1}$ and $g(x) = \sqrt{x+1}$.

15. Find the product of the identity function and the modulus function. Also, find the domain of $f(x) = \sqrt{\frac{1-|x|}{2-|x|}}$.

SOLUTIONS

1. We have, $(2a - 3, b + 2) = (3a + 1, 2b + 5)$
So, $2a - 3 = 3a + 1 \Rightarrow a = -4$
and $b + 2 = 2b + 5 \Rightarrow b = -3$

2. We have, $R = \{(x, y) : y > x \ \forall \ x, y \in U\}$
$\therefore$ $R = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$
So, domain of $R = \{1, 2, 3\}$
and Range of $R = \{2, 3, 4\}$

3. We have, $n(A) = 2$, $n(B) = 2$
$\therefore$ The number of relations from A to $B = 2^{2\times 2} = 2^4$.

4. Given, $A = \{1, 2, 3\}$ and $B = \{4, 5\}$
$A \times B = \{(1, 4), (1, 5), (2, 4), (2, 5), (3, 4), (3, 5)\}$
$B \times A = \{(4, 1), (4, 2), (4, 3), (5, 1), (5, 2), (5, 3)\}$
Now, $(1, 5) \in A \times B$ but $(1, 5) \notin B \times A$.
Hence $A \times B \neq B \times A$.

5. Here $X = \{a, b, c, d\}$ and $Y = \{1, 2, 3, 4, 5\}$.
(i) f_1 is a function because each element of X has unique image, *i.e.*, $f(a) = 1, f(b) = 1, f(c) = 3, f(d) = 4$.
(ii) f_2 is not a function as $a \in X$ has two images, *i.e.*, $f(a) = 1$ and $f(a) = 2$.

6. We have, $A = \{1, 2, 3\}$, $B = \{1, 2, 5, 6\}$
(i) R is the relation 'is less than' from A to B, then
$R = \{(1, 2), (1, 5), (1, 6), (2, 5), (2, 6), (3, 5), (3, 6)\}$
$\therefore$ $R^{-1} = \{(2, 1), (5, 1), (6, 1), (5, 2), (6, 2), (5, 3), (6, 3)\}$
(ii) R is the relation 'is greater than' from A to B, then
$R = \{(2, 1), (3, 1), (3, 2)\}$
$\therefore$ $R^{-1} = \{(1, 2), (1, 3), (2, 3)\}$

7. $f(x) = \sqrt{1-x^2}$
When $1 - x^2 < 0$, $f(x)$ is not defined.
$\therefore$ $f(x)$ is defined when $1 - x^2 \geq 0$
or $-x^2 \geq -1$ or $x^2 \leq 1$
$\therefore$ Domain $= \{x : x \in R, -1 \leq x \leq 1\}$
For range, $y = \sqrt{1-x^2} \Rightarrow y^2 = 1 - x^2$
$\Rightarrow x^2 = 1 - y^2 \Rightarrow x = \sqrt{1-y^2}$,
which is defined when
$1 - y^2 \geq 0$ or $-y^2 \geq -1$
or $y^2 \leq 1$, *i.e.*, $-1 \leq y \leq 1$
But $y \geq 0$
$\therefore$ Range $= \{y : y \in R, 0 \leq y \leq 1\}$.

8. (i) $f(1+h) = \frac{(1+h)^2 - (1+h) + 1}{(1+h)^2 + (1+h) + 1} = \frac{h^2 + h + 1}{h^2 + 3h + 3}$

(ii) $f\left(\frac{\pi}{3}\right) = \frac{\sin(\pi/3) - \cos(\pi/3)}{\sin(\pi/3) + \cos(\pi/3)}$

$$= \frac{\frac{\sqrt{3}}{2} - \frac{1}{2}}{\frac{\sqrt{3}}{2} + \frac{1}{2}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = \frac{(\sqrt{3}-1)^2}{2} = 2 - \sqrt{3}$$

(iii) Since $f(1 + x) = x^2 + 1$, replace x by $1 - x$, we get
$f(1 + 1 - x) = (1 - x)^2 + 1 \Rightarrow f(2 - x) = x^2 - 2x + 2$

(iv) $f(x^2) = \frac{x^2+1}{x^2-1}$ and $[f(x)]^2 = \left(\frac{x+1}{x-1}\right)^2$

9. We have, $f(x) = \frac{x^2}{x^2+1}$
Domain of f: Clearly, $f(x)$ is defined for all $x \in R$ as $x^2 + 1 \neq 0$ for any $x \in R$.
So, Domain $(f) = R$.
Range of f: Let $f(x) = y$. Then,

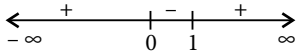
$$y = \frac{x^2}{x^2+1} \Rightarrow x^2 = x^2y + y \Rightarrow x^2(1 - y) = y$$

$$\Rightarrow x^2 = \frac{y}{1-y} \Rightarrow x = \pm\sqrt{\frac{y}{1-y}}$$

Clearly, x will take real values, if

$$\frac{y}{1-y} \geq 0 \Rightarrow \frac{y-0}{y-1} \leq 0$$

$\Rightarrow 0 \leq y < 1$



Hence, range $(f) = [0, 1)$.

10. Let f and g denote respectively the identity function and the reciprocal function. Then, $f : R \to R$ and $g : R - \{0\} \to R$ such that
$f(x) = x$ for all $x \in R$
and $g(x) = \frac{1}{x}$ for all $x \in R - \{0\}$
The domains of f and g are R and $R - \{0\}$ respectively.
Also, we have, $R \cap R - \{0\} = R - \{0\}$
Therefore, $f + g : R - \{0\} \to R$ and $f - g : R - \{0\} \to R$ are given by

$$(f + g)(x) = f(x) + g(x) = x + \frac{1}{x}$$

and $(f - g)(x) = f(x) - g(x) = x - \frac{1}{x}$

11. We have, $f(x) = \frac{1}{\log_{10}(1-x)} + \sqrt{x+2}$
Let, $g(x) = \frac{1}{\log_{10}(1-x)}$ and $h(x) = \sqrt{x+2}$. Then,
$f(x) = g(x) + h(x)$

$\Rightarrow$ Domain (f) = Domain $(g) \cap$ Domain (h)

Now, $g(x) = \dfrac{1}{\log_{10}(1-x)}$ is defined, if

$\log_{10}(1-x)$ is defined and $\log_{10}(1-x) \neq 0$

$\Rightarrow$ $1 - x > 0$ and $1 - x \neq 1 \Rightarrow x < 1$ and $x \neq 0$

$\Rightarrow$ $x \in (-\infty, 0) \cup (0, 1)$

$\therefore$ Domain $(g) = (-\infty, 0) \cup (0, 1)$. ...(i)

$h(x) = \sqrt{x+2}$ is defined, if

$x + 2 \geq 0 \Rightarrow x \geq -2 \Rightarrow x \in [-2, \infty)$.

$\therefore$ Domain $(h) = [-2, \infty)$ ...(ii)

Domain of f is intersection of (i) and (ii)

Hence, Domain $(f) = [-2, 0) \cup (0, 1)$.

12. (i) We have, $f(x) = |x - 3|$

Clearly, $f(x)$ is defined for all $x \in R$

$\therefore$ Domain $(f) = R$.

Now, $|x - 3| \geq 0$ for all $x \in R$

$\Rightarrow$ $0 \leq |x - 3| < \infty$ for all $x \in R$

$\Rightarrow$ $0 \leq f(x) < \infty$ for all $x \in R$

$\Rightarrow$ $f(x) \in [0, \infty)$ for all $x \in R$

Hence, Range $(f) = [0, \infty)$.

(ii) We have, $f(x) = 1 - |x - 2|$

We observe that $f(x)$ is defined for all $x \in R$.

$\therefore$ Domain $(f) = R$.

Now, $0 \leq |x - 2| < \infty$ for all $x \in R$

$\Rightarrow$ $-\infty < -|x - 2| \leq 0$ for all $x \in R$

$\Rightarrow$ $-\infty < 1 - |x - 2| \leq 1$ for all $x \in R$

$\Rightarrow$ $-\infty < f(x) \leq 1$ for all $x \in R$

$\Rightarrow$ $f(x) \in (-\infty, 1]$

Hence, Range $(f) = (-\infty, 1]$

(iii) We have, $f(x) = \dfrac{|x-4|}{x-4}$

Clearly, $f(x)$ is defined for all $x \in R$ except at $x = 4$.

$\therefore$ Domain $(f) = R - \{4\}$

Now, $f(x) = \dfrac{|x-4|}{x-4}$

$$f(x) = \begin{cases} \dfrac{x-4}{x-4} = 1, & \text{if } x > 4 \\ \dfrac{-(x-4)}{x-4} = -1, & \text{if } x < 4 \end{cases}$$

$\therefore$ Range $(f) = \{-1, 1\}$

13. (i) For any $x \in Z$, we have

$x - x = 0 = 0 \times n \Rightarrow x - x$ is divisible by n.

$\Rightarrow$ $(x, x) \in R$

Thus, $(x, x) \in R$ for all $x \in Z$.

(ii) Let $(x, y) \in R$. Then, $x - y$ is divisible by n

$\Rightarrow$ $x - y = \lambda n$ for some $\lambda \in Z \Rightarrow y - x = (-\lambda)n$

$\Rightarrow$ $y - x$ is divisible by n $[\because \lambda \in Z \Rightarrow -\lambda \in Z]$

$\Rightarrow$ $(y, x) \in R$

Thus, $(x, y) \in R \Rightarrow (y, x) \in R$ for all $x, y \in Z$.

(iii) Let $(x, y) \in R$ and $(y, z) \in R$. Then,

$(x, y) \in R \Rightarrow x - y$ is divisible by $n \Rightarrow x - y = \lambda n$ for some $\lambda \in Z$

$(y, z) \in R \Rightarrow y - z$ is divisible by $n \Rightarrow y - z = \mu n$ for some $\mu \in Z$

$\therefore$ $(x, y) \in R$ and $(y, z) \in R$

$\Rightarrow$ $x - y = \lambda n$ and $y - z = \mu n$

$\Rightarrow$ $(x - y) + (y - z) = \lambda n + \mu n \Rightarrow x - z = (\lambda + \mu)n$

$\Rightarrow$ $x - z$ is divisible by n. $[\because (\lambda + \mu) \in Z]$

$\Rightarrow$ $(x, z) \in R$

Thus, $(x, y) \in R$ and $(y, z) \in R \Rightarrow (x, z) \in R$.

14. (i) $f + g : R \to R$ given by $(f + g)(x) = f(x) + g(x)$
$= x^3 + x + 2$

$cf : R \to R$ given by $(cf)(x) = cf(x) = c(x^3 + 1)$

$\dfrac{f}{g} : R - \{-1\} \to R$ given by

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} = x^2 - x + 1$$

(ii) $f + g : [1, \infty) \to R$ defined by

$(f + g)(x) = \sqrt{x-1} + \sqrt{x+1}$

$cf : [1, \infty) \to R$ defined by $(cf)(x) = c\sqrt{x-1}$

$\dfrac{f}{g} : [1, \infty) \to R$ defined by $\left(\dfrac{f}{g}\right)(x) = \sqrt{\dfrac{x-1}{x+1}}$

15. Let f and g denote respectively the identity function and modulus function.

Then, $f : R \to R$ such that $f(x) = x$ for all x

and, $g : R \to R$ such that $g(x) = |x|$ for all x.

Clearly, f and g have the same domain. Therefore, the product fg is a function from R to itself and is given by

$(fg)(x) = f(x)\, g(x) \Rightarrow (fg)(x) = x|x|$

$$\Rightarrow (fg)(x) = \begin{cases} x^2, & \text{if } x \geq 0 \\ -x^2, & \text{if } x < 0. \end{cases}$$

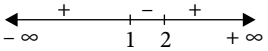
We have, $f(x) = \sqrt{\dfrac{1-|x|}{2-|x|}}$

We observe that $f(x)$ is defined for all x satisfying

$\dfrac{1-|x|}{2-|x|} \geq 0 \Rightarrow \dfrac{|x|-1}{|x|-2} \geq 0$

$\Rightarrow$ $|x| \leq 1$ or, $|x| > 2$

$\Rightarrow$ $x \in [-1, 1]$



or, $x \in (-\infty, -2) \cup (2, \infty)$

$\Rightarrow$ $x \in (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$

Hence, domain $(f) = (-\infty, -2) \cup (2, \infty) \cup [-1, 1]$.

Series 2

ACE YOUR WAY CBSE XII

INVERSE TRIGONOMETRIC FUNCTIONS

INVERSE OF A FUNCTION

If $f : X \to Y$ is one-one and onto, then there exists a unique function $g : Y \to X$ s.t. $g(y) = x$, where $x \in X$ is s.t. $y = f(x)$. Thus, the domain of g = range of f and range of g = domain of f. The function g is called the inverse of f and is denoted as f^{-1}.

Thus $f(x) = y \Leftrightarrow f^{-1}(y) = x$ *i.e.*, image of x under f is $y \Leftrightarrow$ image of y under f^{-1} is x

Also, $g^{-1} = (f^{-1})^{-1} = f$

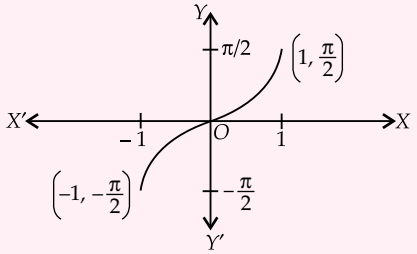
$f^{-1}\{f(x)\} = f^{-1}(y) = x \ \forall \ x \in X$ and $f\{f^{-1}(y)\} = f(x) = y \ \forall \ x \in X$

DOMAIN AND RANGE OF INVERSE TRIGONOMETRIC FUNCTIONS

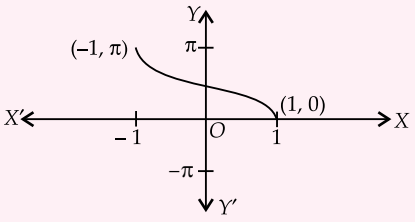
	Functions	Domain	Principal value branch
1.	$y = \sin^{-1} x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
2.	$y = \cos^{-1} x$	$-1 \le x \le 1$	$0 \le y \le \pi$
3.	$y = \tan^{-1} x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
4.	$y = \cot^{-1} x$	$-\infty < x < \infty$	$0 < y < \pi$
5.	$y = \sec^{-1} x$	$x \le -1$ or $x \ge 1$	$0 \le y \le \pi, \ y \ne \frac{\pi}{2}$
6.	$y = \operatorname{cosec}^{-1} x$	$x \le -1$ or $x \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \ y \ne 0$

GRAPHS OF INVERSE TRIGONOMETRIC FUNCTIONS

1. $y = \sin^{-1}x$
Domain = $[-1, 1]$
Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$



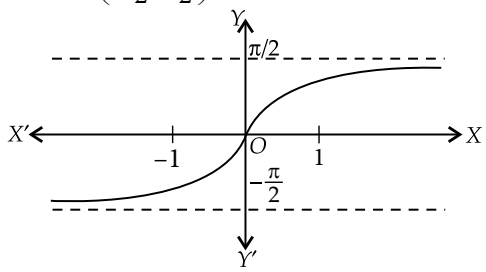
2. $y = \cos^{-1}x$
Domain = $[-1, 1]$
Range = $[0, \pi]$



3. $y = \tan^{-1}x$

Domain = $(-\infty, \infty)$

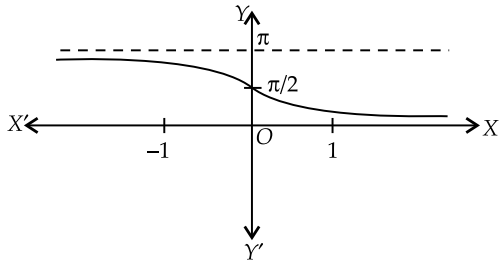
Range = $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$



4. $y = \cot^{-1}x$

Domain = $(-\infty, \infty)$

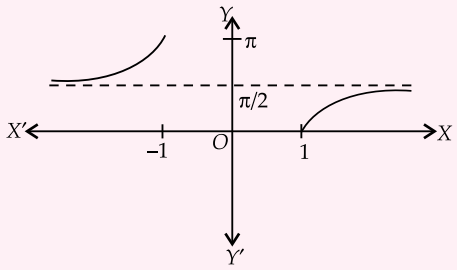
Range = $(0, \pi)$



5. $y = \sec^{-1}x$

Domain = $(-\infty, -1] \cup [1, \infty)$

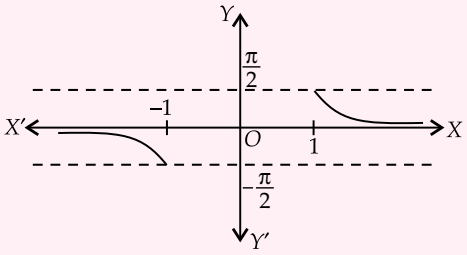
Range = $[0, \pi] - \left\{\frac{\pi}{2}\right\}$



6. $y = \text{cosec}^{-1}x$

Domain = $(-\infty, -1] \cup [1, \infty)$

Range = $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$



PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTIONS

1.	• $\sin^{-1}(\sin\theta) = \theta, \forall\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$	2.	• $\sin(\sin^{-1}x) = x, \forall\ x \in [-1, 1]$
	• $\cos^{-1}(\cos\theta) = \theta, \forall\ \theta \in [0, \pi]$		• $\cos(\cos^{-1}x) = x, \forall\ x \in [-1, 1]$
	• $\tan^{-1}(\tan\theta) = \theta, \forall\ \theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$		• $\tan(\tan^{-1}x) = x, \forall\ x \in R$
	• $\cot^{-1}(\cot\theta) = \theta, \forall\ \theta \in (0, \pi)$		• $\cot(\cot^{-1}x) = x, \forall\ x \in R$
	• $\sec^{-1}(\sec\theta) = \theta, \forall\ \theta \in [0, \pi] - \left\{\frac{\pi}{2}\right\}$		• $\sec(\sec^{-1}x) = x, \forall\ x \in R - (-1, 1)$
	• $\text{cosec}^{-1}(\text{cosec}\ \theta) = \theta, \forall\ \theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$		• $\text{cosec}(\text{cosec}^{-1}x) = x, \forall\ x \in R - (-1, 1)$

3.	• $\sin^{-1}(-x) = -\sin^{-1}(x), \forall\ x \in [-1, 1]$	• $\cot^{-1}(-x) = \pi - \cot^{-1}x, \forall\ x \in R$
	• $\cos^{-1}(-x) = \pi - \cos^{-1}x, \forall\ x \in [-1, 1]$	• $\sec^{-1}(-x) = \pi - \sec^{-1}x, \forall\ x \in (-\infty, -1] \cup [1, \infty)$
	• $\tan^{-1}(-x) = -\tan^{-1}x, \forall\ x \in R$	• $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x, \forall\ x \in (-\infty, -1] \cup [1, \infty)$

4.	• $\sin^{-1}(1/x) = \text{cosec}^{-1}x, \forall\ x \in (-\infty, -1] \cup [1, \infty)$	5.	• $\sin^{-1}x + \cos^{-1}x = \pi/2, \forall\ x \in [-1, 1]$
	• $\cos^{-1}(1/x) = \sec^{-1}x, \forall\ x \in (-\infty, -1] \cup [1, \infty)$		• $\tan^{-1}x + \cot^{-1}x = \pi/2, \forall\ x \in R$
	• $\tan^{-1}(1/x) = \begin{cases} \cot^{-1}x & , \text{ for } x > 0 \\ -\pi + \cot^{-1}x & , \text{ for } x < 0 \end{cases}$		• $\sec^{-1}x + \text{cosec}^{-1}x = \pi/2, \forall\ x \in (-\infty, -1] \cup [1, \infty)$

6.

- $\tan^{-1}x+\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & , \text{ if } x>0,\ y>0,\ xy<1\\ \pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & , \text{ if } x>0,\ y>0 \text{ and } xy>1\\ -\pi+\tan^{-1}\left(\dfrac{x+y}{1-xy}\right) & , \text{ if } x<0,\ y<0 \text{ and } xy>1\end{cases}$

- $\tan^{-1}x-\tan^{-1}y=\begin{cases}\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & , \text{ if } x>0,\ y<0,\ xy>-1\\ \pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & , \text{ if } x>0,\ y<0 \text{ and } xy<-1\\ -\pi+\tan^{-1}\left(\dfrac{x-y}{1+xy}\right) & , \text{ if } x<0,\ y>0 \text{ and } xy<-1\end{cases}$

7.

- $\sin^{-1}x+\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy<0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & , \text{ if } 0<x,\ y\le 1 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}+y\sqrt{1-x^2}\right\} & , \text{ if } -1\le x,\ y<0 \text{ and } x^2+y^2>1\end{cases}$

- $\sin^{-1}x-\sin^{-1}y=\begin{cases}\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x^2+y^2\le 1 \text{ or if } xy>0 \text{ and } x^2+y^2>1\\ \pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & , \text{ if } 0<x\le 1,\ -1\le y\le 0 \text{ and } x^2+y^2>1\\ -\pi-\sin^{-1}\left\{x\sqrt{1-y^2}-y\sqrt{1-x^2}\right\} & , \text{ if } -1\le x<0,\ 0<y\le 1 \text{ and } x^2+y^2>1\end{cases}$

8.

- $\cos^{-1}x+\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x+y\ge 0\\ 2\pi-\cos^{-1}\left\{xy-\sqrt{1-x^2}\sqrt{1-y^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x+y\le 0\end{cases}$

- $\cos^{-1}x-\cos^{-1}y=\begin{cases}\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} & , \text{ if } -1\le x,\ y\le 1 \text{ and } x\le y\\ -\cos^{-1}\left\{xy+\sqrt{1-x^2}\sqrt{1-y^2}\right\} & , \text{ if } -1\le y\le 0,\ 0<x\le 1 \text{ and } x\ge y\end{cases}$

9.

- $2\sin^{-1}x=\begin{cases}\sin^{-1}(2x\sqrt{1-x^2}) & , \text{ if } -\dfrac{1}{\sqrt{2}}\le x\le\dfrac{1}{\sqrt{2}}\\ \pi-\sin^{-1}(2x\sqrt{1-x^2}) & , \text{ if } \dfrac{1}{\sqrt{2}}\le x\le 1\\ -\pi-\sin^{-1}(2x\sqrt{1-x^2}) & , \text{ if } -1\le x\le-\dfrac{1}{\sqrt{2}}\end{cases}$

	• $3\sin^{-1}x=\begin{cases}\sin^{-1}(3x-4x^3) & , \text{ if } -\frac{1}{2}\le x\le\frac{1}{2}\\ \pi-\sin^{-1}(3x-4x^3) & , \text{ if } \frac{1}{2}<x\le 1\\ -\pi-\sin^{-1}(3x-4x^3), & \text{if } -1\le x<-\frac{1}{2}\end{cases}$	
10.	• $2\cos^{-1}x=\begin{cases}\cos^{-1}(2x^2-1) & , \text{ if } 0\le x\le 1\\ 2\pi-\cos^{-1}(2x^2-1), & \text{if } -1\le x\le 0\end{cases}$	• $3\cos^{-1}x=\begin{cases}\cos^{-1}(4x^3-3x) & , \text{ if } \frac{1}{2}\le x\le 1\\ 2\pi-\cos^{-1}(4x^3-3x), & \text{if } -\frac{1}{2}\le x\le\frac{1}{2}\\ 2\pi+\cos^{-1}(4x^3-3x), & \text{if } -1\le x\le-\frac{1}{2}\end{cases}$
11.	• $2\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{2x}{1-x^2}\right) & , \text{ if } -1<x<1\\ \pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right) & , \text{ if } x>1\\ -\pi+\tan^{-1}\left(\frac{2x}{1-x^2}\right), & \text{if } x<-1\end{cases}$	• $3\tan^{-1}x=\begin{cases}\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) & , \text{ if } -\frac{1}{\sqrt{3}}<x<\frac{1}{\sqrt{3}}\\ \pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right) & , \text{ if } x>\frac{1}{\sqrt{3}}\\ -\pi+\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right), & \text{if } x<-\frac{1}{\sqrt{3}}\end{cases}$
12.	• $2\tan^{-1}x=\begin{cases}\sin^{-1}\left(\frac{2x}{1+x^2}\right) & , \text{if } -1\le x\le 1\\ \pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right) & , \text{if } x>1\\ -\pi-\sin^{-1}\left(\frac{2x}{1+x^2}\right), & \text{if } x<-1\end{cases}$	• $2\tan^{-1}x=\begin{cases}\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) & , \text{ if } 0\le x<\infty\\ -\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right), & \text{if } -\infty<x\le 0\end{cases}$

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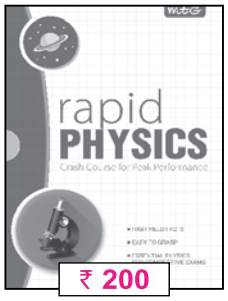

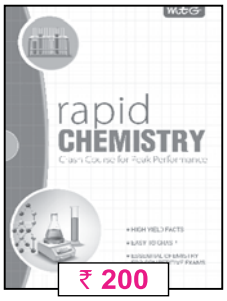

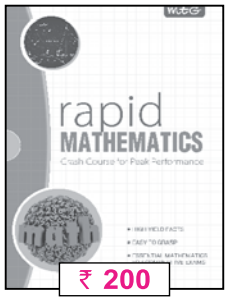

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13.

- $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)=\text{cosec}^{-1}\left(\frac{1}{x}\right)$
- $\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right)=\cot^{-1}\frac{x}{\sqrt{1-x^2}}=\sec^{-1}\frac{1}{x}=\text{cosec}^{-1}\left(\frac{1}{\sqrt{1-x^2}}\right)$
- $\tan^{-1}x=\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)=\cos^{-1}\left(\frac{1}{\sqrt{1+x^2}}\right)=\cot^{-1}\left(\frac{1}{x}\right)=\sec^{-1}\sqrt{1+x^2}=\text{cosec}^{-1}\left(\frac{\sqrt{1+x^2}}{x}\right)$

14.

- If $x_1, x_2, \ldots, x_n \in R$, then
$\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n=\tan^{-1}\left(\frac{S_1-S_3+S_5-S_7+\ldots}{1-S_2+S_4-S_6+\ldots}\right)$
where S_k = Sum of the products of $x_1, x_2, \ldots, x_n$ taken k at a time.

Very Short Answer Type

1. Find the value of $\sin\left[\tan^{-1}(-\sqrt{3})+\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$.
2. Evaluate : $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
3. Find the principal value of $\cos^{-1}\left(-\frac{1}{2}\right)$.
4. Prove that : $\tan^{-1}\left(\frac{m}{n}\right)-\tan^{-1}\left(\frac{m-n}{m+n}\right)=\frac{\pi}{4}$
5. Find the value of $\cot\left(\frac{\pi}{4}-2\cot^{-1}3\right)$.

Short Answer Type

6. Find the principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$.
7. Prove that :
$\cot(\cot^{-1}7+\cot^{-1}8+\cot^{-1}18)=3$
8. If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\frac{\pi}{2}$, prove that
$xy+yz+zx=1$.
9. Solve :
$\tan^{-1}\left(\frac{1}{1+2x}\right)+\tan^{-1}\left(\frac{1}{4x+1}\right)=\tan^{-1}\left(\frac{2}{x^2}\right)$
10. Prove that : $\tan^{-1}\left(\frac{1}{4}\right)+\tan^{-1}\left(\frac{2}{9}\right)=\frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)$

Long Answer Type

11. Prove that :
$4\tan^{-1}\left(\frac{1}{5}\right)-\tan^{-1}\left(\frac{1}{70}\right)+\tan^{-1}\left(\frac{1}{99}\right)=\frac{\pi}{4}$
12. Let $\cos^{-1}\left(\frac{x}{a}\right)+\cos^{-1}\left(\frac{y}{b}\right)=\alpha$.
Prove that $\frac{x^2}{a^2}-\frac{2xy}{ab}(\cos\alpha)+\frac{y^2}{b^2}=\sin^2\alpha$.
Also, write $\cos^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$ in simplest form.
13. (i) Solve : $\tan^{-1}\left(\frac{x-1}{x-2}\right)+\tan^{-1}\left(\frac{x+1}{x+2}\right)=\frac{\pi}{4}$
(ii) Prove that : $\sec^2(\tan^{-1}2)+\text{cosec}^2(\cot^{-1}3)=15$
14. Solve for x : $\sin^{-1}(1-x)-2\sin^{-1}x=\frac{\pi}{2}$.
Also, evaluate $\sin(3\sin^{-1}0.4)$.

SOLUTIONS

1. We have, $\sin\left[\tan^{-1}(-\sqrt{3})+\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
$=\sin\left[-\frac{\pi}{3}+\pi-\frac{\pi}{6}\right]=\sin\left(\frac{\pi}{2}\right)=1$

2. $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$
$=\sin\left[\frac{\pi}{2}-\left(-\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right)\right]=\sin\left[\frac{\pi}{2}-\left(\frac{-\pi}{3}\right)\right]$
$=\sin\frac{5\pi}{6}=\frac{1}{2}$

3. Let principal value of $\cos^{-1}\left(-\frac{1}{2}\right)=\theta$,
then $\cos\theta=-\frac{1}{2}$.

Since $0 \le \cos^{-1}x \le \pi$, $\therefore\ 0 \le \theta \le \pi$

Also $\cos\theta$ is negative, $\therefore\ \dfrac{\pi}{2} < \theta < \pi$

Now $\cos\theta = \dfrac{-1}{2} = -\cos\dfrac{\pi}{3} = \cos\left(\pi - \dfrac{\pi}{3}\right) = \cos\dfrac{2\pi}{3}$

$\therefore\ \theta = \dfrac{2\pi}{3}$.

4 $\text{L.H.S.} = \tan^{-1}\dfrac{m}{n} - \tan^{-1}\dfrac{m-n}{m+n} = \tan^{-1}\left(\dfrac{\frac{m}{n} - \frac{m-n}{m+n}}{1 + \frac{m}{n}\times\frac{m-n}{m+n}}\right)$

$= \tan^{-1}\left(\dfrac{m^2+n^2}{m^2+n^2}\right) = \tan^{-1}(1) = \dfrac{\pi}{4} = \text{R.H.S.}$

5. We have,

$\cot\left(\dfrac{\pi}{4} - 2\cot^{-1}3\right) = \cot\left[\dfrac{\pi}{4} - \cot^{-1}\left(\dfrac{3^2-1}{2\times 3}\right)\right]$

$= \cot\left(\dfrac{\pi}{4} - \cot^{-1}\dfrac{4}{3}\right) = \dfrac{\cot\frac{\pi}{4}\cdot\frac{4}{3}+1}{\frac{4}{3} - \cot\frac{\pi}{4}} = \dfrac{\frac{4}{3}+1}{\frac{4}{3}-1} = 7$

6. $\sin\dfrac{2\pi}{3} = \sin\left(\pi - \dfrac{\pi}{3}\right) = \sin\dfrac{\pi}{3} = \dfrac{\sqrt{3}}{2}$.

$\therefore\ \sin^{-1}\left(\sin\dfrac{2\pi}{3}\right) = \sin^{-1}\left(\dfrac{\sqrt{3}}{2}\right) = \theta\,(\text{say}).$

$\therefore\ \sin\theta = \dfrac{\sqrt{3}}{2}$.

Since the principal value of $\sin^{-1}x$ is given by

$-\dfrac{\pi}{2} \le \sin^{-1}x \le \dfrac{\pi}{2},\ \ \therefore\ -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

Also, $\sin\theta$ is positive $\therefore\ 0 \le \theta \le \dfrac{\pi}{2}$

Now $\sin\theta = \dfrac{\sqrt{3}}{2} = \sin\dfrac{\pi}{3}\ \ \therefore\ \theta = \dfrac{\pi}{3}$

Hence, the principal value of $\sin^{-1}\left(\sin\dfrac{2\pi}{3}\right)$ is $\dfrac{\pi}{3}$.

7. We have, $\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18$

$= \tan^{-1}\left(\dfrac{1}{7}\right) + \tan^{-1}\left(\dfrac{1}{8}\right) + \tan^{-1}\left(\dfrac{1}{18}\right)$

$= \tan^{-1}\left(\dfrac{(1/7)+(1/8)}{1-(1/7)(1/8)}\right) + \tan^{-1}\left(\dfrac{1}{18}\right)$

$= \tan^{-1}\left(\dfrac{3}{11}\right) + \tan^{-1}\left(\dfrac{1}{18}\right)$

$= \tan^{-1}\left(\dfrac{(3/11)+(1/18)}{1-(3/11)(1/18)}\right) = \tan^{-1}\left(\dfrac{1}{3}\right) = \cot^{-1}3$

Thus, $\cot(\cot^{-1}7 + \cot^{-1}8 + \cot^{-1}18) = 3$

8. We have, $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \dfrac{\pi}{2}$

$\Rightarrow\ \tan^{-1}x + \tan^{-1}y = \dfrac{\pi}{2} - \tan^{-1}z$

$\Rightarrow\ \tan^{-1}\left(\dfrac{x+y}{1-xy}\right) = \cot^{-1}z = \tan^{-1}\left(\dfrac{1}{z}\right)$

$\Rightarrow\ \dfrac{x+y}{1-xy} = \dfrac{1}{z} \Rightarrow zx + yz = 1 - xy$

$\Rightarrow\ xy + yz + zx = 1$

9. We have,

$\tan^{-1}\left(\dfrac{1}{1+2x}\right) + \tan^{-1}\left(\dfrac{1}{4x+1}\right) = \tan^{-1}\left(\dfrac{2}{x^2}\right)$

$\Rightarrow\ \tan^{-1}\left[\dfrac{\frac{1}{1+2x}+\frac{1}{4x+1}}{1-\frac{1}{1+2x}\times\frac{1}{4x+1}}\right] = \tan^{-1}\left(\dfrac{2}{x^2}\right)$

$\Rightarrow\ \dfrac{(4x+1)+(1+2x)}{(1+2x)(4x+1)-1} = \dfrac{2}{x^2} \Rightarrow \dfrac{6x+2}{8x^2+6x} = \dfrac{2}{x^2}$

$\Rightarrow\ 2(3x+1)x^2 = 2(8x^2+6x) \Rightarrow 3x^3 + x^2 = 8x^2 + 6x$

$\Rightarrow\ 3x^3 - 7x^2 - 6x = 0 \Rightarrow x(x-3)(3x+2) = 0$

Hence $x = 0,\ 3,\ -\dfrac{2}{3}$

10. $\text{L.H.S.} = \tan^{-1}\left(\dfrac{1}{4}\right) + \tan^{-1}\left(\dfrac{2}{9}\right)$

$= \tan^{-1}\left[\dfrac{\frac{1}{4}+\frac{2}{9}}{1-\left(\frac{1}{4}\right)\left(\frac{2}{9}\right)}\right] = \tan^{-1}\left(\dfrac{17}{34}\right) = \tan^{-1}\left(\dfrac{1}{2}\right)$

$= \dfrac{1}{2}\tan^{-1}\left[\dfrac{2\left(\frac{1}{2}\right)}{1-\left(\frac{1}{2}\right)^2}\right] = \dfrac{1}{2}\tan^{-1}\dfrac{1}{(3/4)} = \dfrac{1}{2}\tan^{-1}\left(\dfrac{4}{3}\right) = \text{R.H.S.}$

11. We have,

$4\tan^{-1}\left(\dfrac{1}{5}\right) = \dfrac{\pi}{4} + \tan^{-1}\left(\dfrac{1}{70}\right) - \tan^{-1}\left(\dfrac{1}{99}\right)$

Now, $\text{L.H.S.} = 4\tan^{-1}\left(\dfrac{1}{5}\right) = 2\cdot 2\tan^{-1}\left(\dfrac{1}{5}\right)$

$= 2\tan^{-1}\left[\dfrac{2\cdot\frac{1}{5}}{1-\frac{1}{25}}\right] = 2\tan^{-1}\left(\dfrac{5}{12}\right) = \tan^{-1}\left[\dfrac{2\cdot\frac{5}{12}}{1-\frac{25}{144}}\right]$

$= \tan^{-1}\left(\frac{120}{119}\right)$

$\text{R.H.S.} = \frac{\pi}{4} + \tan^{-1}\left(\frac{1}{70}\right) - \tan^{-1}\left(\frac{1}{99}\right)$

$= \frac{\pi}{4} + \tan^{-1}\left[\frac{\frac{1}{70} - \frac{1}{99}}{1 + \frac{1}{70}\cdot\frac{1}{99}}\right]$

$= \frac{\pi}{4} + \tan^{-1}\left(\frac{1}{239}\right) = \tan^{-1}1 + \tan^{-1}\left(\frac{1}{239}\right)$

$= \tan^{-1}\left[\frac{1 + \frac{1}{239}}{1 - \frac{1}{239}}\right] = \tan^{-1}\left(\frac{240}{238}\right) = \tan^{-1}\left(\frac{120}{119}\right)$

$\therefore$ L.H.S. = R.H.S.

12. Given as $\cos^{-1}\left(\frac{x}{a}\right) + \cos^{-1}\left(\frac{y}{b}\right) = \alpha$

$\Rightarrow \cos^{-1}\left(\frac{x}{a}\cdot\frac{y}{b} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}}\right) = \alpha$

$\Rightarrow \frac{x}{a}\cdot\frac{y}{b} - \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} = \cos\alpha$

$\Rightarrow \left[\frac{x}{a}\cdot\frac{y}{b} - \cos\alpha\right]^2 = \left(1-\frac{x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)$

$\Rightarrow \frac{x^2}{a^2}\cdot\frac{y^2}{b^2} + \cos^2\alpha - \frac{2xy}{ab}(\cos\alpha) = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{x^2}{a^2}\cdot\frac{y^2}{b^2}$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy}{ab}(\cos\alpha) = 1 - \cos^2\alpha$

$\Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{2xy(\cos\alpha)}{ab} = \sin^2\alpha$ (Hence proved)

Now, $\cos^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$

$\left[\text{Put } x = a\cot\theta \Rightarrow \theta = \cot^{-1}\left(\frac{x}{a}\right)\right]$

$= \cos^{-1}\left(\frac{a\cot\theta}{\sqrt{a^2\cot^2\theta + a^2}}\right) = \cos^{-1}\left(\frac{a\cot\theta}{a\,\text{cosec}\,\theta}\right)$

$= \cos^{-1}(\cos\theta) = \theta = \cot^{-1}\left(\frac{x}{a}\right)$

Hence, simplest form of $\cos^{-1}\left(\frac{x}{\sqrt{x^2+a^2}}\right)$ is $\cot^{-1}\left(\frac{x}{a}\right)$.

13. (i) We have, $\tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \frac{\pi}{4}$

$\Rightarrow \tan^{-1}\left(\frac{x-1}{x-2}\right) + \tan^{-1}\left(\frac{x+1}{x+2}\right) = \tan^{-1}1$

$\Rightarrow \tan^{-1}\left(\frac{x-1}{x-2}\right) = \tan^{-1}1 - \tan^{-1}\left(\frac{x+1}{x+2}\right)$

$\Rightarrow \tan^{-1}\left(\frac{x-1}{x-2}\right) = \tan^{-1}\left(\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}\right)$

$\Rightarrow \tan^{-1}\left(\frac{x-1}{x-2}\right) = \tan^{-1}\left(\frac{1}{2x+3}\right)$

$\Rightarrow \frac{x-1}{x-2} = \frac{1}{2x+3} \Rightarrow (2x+3)(x-1) = x-2$

$\Rightarrow 2x^2 - 1 = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$

(ii) L.H.S. = $\sec^2(\tan^{-1}2) + \text{cosec}^2(\cot^{-1}3)$

$= \left\{\sec(\tan^{-1}2)\right\}^2 + \left\{\text{cosec}(\cot^{-1}3)\right\}^2$

$= \left\{\sec\left(\tan^{-1}\frac{2}{1}\right)\right\}^2 + \left\{\text{cosec}\left(\cot^{-1}\frac{3}{1}\right)\right\}^2$

$= \left\{\sec(\sec^{-1}\sqrt{5})\right\}^2 + \left\{\text{cosec}(\text{cosec}^{-1}\sqrt{10})\right\}^2$

$= (\sqrt{5})^2 + (\sqrt{10})^2 = 15 = \text{R.H.S.}$

14. We have, $\sin^{-1}(1-x) - 2\sin^{-1}x = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}(1-x) = \frac{\pi}{2} + 2\sin^{-1}x$

$\Rightarrow 1 - x = \sin(\pi/2 + 2\sin^{-1}x)$

$\Rightarrow 1 - x = \cos(2\sin^{-1}x)$

$\Rightarrow 1 - x = \cos\{\cos^{-1}(1 - 2x^2)\}$

$\Rightarrow 1 - x = (1 - 2x^2)$

$\Rightarrow x(2x - 1) = 0 \Rightarrow x = 0, \frac{1}{2}$

For, $x = \frac{1}{2}$, we have

L.H.S = $\sin^{-1}(1 - x) - 2\sin^{-1}x$

$= \sin^{-1}\frac{1}{2} - 2\sin^{-1}\frac{1}{2} = -\sin^{-1}\frac{1}{2} = -\frac{\pi}{6} \neq$ R.H.S

So, $x = 1/2$ is not a root of the given equation.

Clearly, $x = 0$ satisfies the equation.

Hence, $x = 0$ is a root of the given equation.

We have, $\sin(3\sin^{-1}0.4)$

$= \sin[\sin^{-1}\{3 \times 0.4 - 4 \times (0.4)^3\}]$

$= \sin[\sin^{-1}\{(1.2) - 0.256\}] = 0.944$

JEE WORK GUTS

SECTION-I

SINGLE CORRECT OPTION

This section contains 7 multiple choice questions. Each question has four choices (a), (b), (c) and (d) out of which ONLY ONE is correct. [3 marks for correct answer and –1 for wrong answer]

1. If $H_n = 1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+.....+\frac{1}{n}$ and $H_n' = \frac{n+1}{2}-\left[\frac{1}{n(n-1)}+\frac{2}{(n-1)(n-2)}+\frac{3}{(n-2)(n-3)}+...+\frac{n-2}{3\cdot 2}\right]$, then

(a) $H_n = H_n'$ (b) $2H_n = H_n'$
(c) $H_n = 2H_n'$ (d) $H_n = 3H_n'$

2. If $f(x) = \sin x - x$, then $\int_{-2\pi}^{2\pi} \left|f^{-1}(x)\right| dx$ equals

(a) π^2 (b) $2\pi^2$ (c) $3\pi^2$ (d) $4\pi^2$

3. Given two circles $x^2+y^2+3\sqrt{2}(x+y)=0$ and $x^2+y^2+5\sqrt{2}(x+y)=0$. Let the radius of the third circle, which is tangent to the two given circles and to their common diameter, be $\frac{2p-1}{p}$. The value of p is

(a) 7 (b) 5 (c) 8 (d) 10

4. The equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$, is

(a) $(x-1)^2+(y-0)^2=\frac{11}{3}$

(b) $(x-1)^2+(y-0)^2=\frac{22}{3}$

(c) $(x-1)^2+(y-0)^2=11$

(d) $(x+1)^2+(y+0)^2=\frac{11}{3}$

5. If $z = x + 3i$, then the value of $\int_2^4 \left[\arg\left|\frac{z-i}{z+i}\right|\right] dx$, (where [.] denotes the greatest integer function and $i = \sqrt{-1}$), is

(a) $3\sqrt{2}$ (b) $6\sqrt{3}$
(c) $\sqrt{6}$ (d) None of these

6. Let $\overrightarrow{AB} = 3\hat{i}+\hat{j}-\hat{k}$ and $\overrightarrow{AC} = \hat{i}-\hat{j}+3\hat{k}$. If the point P on the line segment BC is equidistant from AB and AC then $\overrightarrow{AP}$ is

(a) $2\hat{i}-\hat{k}$ (b) $\hat{i}-2\hat{k}$
(c) $2\hat{i}+\hat{k}$ (d) None of these

7. If $f(n) = \alpha^n + \beta^n$ and $\begin{vmatrix} 3 & 1+f(1) & 1+f(2) \\ 1+f(1) & 1+f(2) & 1+f(3) \\ 1+f(2) & 1+f(3) & 1+f(4) \end{vmatrix} = k(1-\alpha)^2(1-\beta)^2(\alpha-\beta)^2$, then k is equal to

(a) 1 (b) –1 (c) $\alpha\beta$ (d) $\frac{1}{\alpha\beta}$

SECTION-II

MULTIPLE CORRECT OPTION

This section contains 4 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) out of which ONE OR MORE is/are correct. [4 marks for correct answer and 0 for wrong answer]

8. If $f(x)$ is a cubic polynomial such that $f(0) + 2 = 0$ and $f'''(0) = 6$ and $f(x) = 0$ possess three positive integral roots then

(a) $f'(1) = 0$ (b) $2f'(0) + f''(0) = 0$
(c) $2f'(0) + f''(0) = 2$ (d) $f'(2) = 0$

9. On the interval $A = [-2, 2]$, the function

$$f(x) = \begin{cases} (x+1)e^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

(a) is continuous for all $x \in A - \{0\}$
(b) is continuous for all $x \in A$
(c) assumes all intermediate values from $f(-2)$ to $f(2)$
(d) has maximum value equal to $3/e$

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10. The number of real solutions of the equation $3\sqrt{[32\cos^6 x - 48\cos^4 x + 18\cos^2 x - 1]^2} = \sin x$ is

(a) 6, $x \in \left[0, \frac{\pi}{2}\right]$ (b) 12, $x \in [0, \pi]$

(c) 6, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (d) 12, $x \in [-\pi, \pi]$

11. Let $f(x) = ax^2 + bx + c$, where $a, b, c \in R$. Suppose $|f(x)| \le 1 \ \forall \ x \in [0, 1]$, then

(a) $|a| \le 8$ (b) $|b| \le 8$

(c) $|c| \le 1$ (d) $|a| + |b| + |c| \le 17$

SECTION-III

LINKED COMPREHENSION TYPE

This section contains 2 paragraphs. Based upon one of the paragraphs 2 multiple choice questions and based on the other paragraph 3 multiple choice questions have to be answered. Each of these questions has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct. [3 marks for correct answer and –1 for wrong answer]

Paragraph for Question Nos. 12 and 13

Consider $a_1 + \frac{1}{a_2} + \frac{1}{2a_3} + ... + \frac{1}{(n-1)a_n} = \frac{1}{2a_{n+1}}$; $n \ge 3$, $n \in N$ where $a_1 = 1$, $a_2 = 1/2$.

12. $\sum_{i=1}^{\infty} a_i$ is

(a) 1/2 (b) 1 (c) 2 (d) 3/2

13. $\frac{1}{a_2} + \frac{1}{2a_3} + \frac{1}{3a_4} + ... + \frac{1}{(n-1)a_n} - \frac{1}{3a_3} - \frac{1}{4a_4} - ... - \frac{1}{na_n}$ is equal to

(a) 0 (b) n (c) $n - 1$ (d) $n + 1$

Paragraph for Question Nos. 14 to 16

Two persons A and B are playing with 5 dice. Player A throws 3 dice and Player B throws 2 dice. The trials go on simultaneously and successively until 6 shows on at least one of the dice.

14. $P(A)$: The probability that player A and not B first obtains 6 is

(a) $\frac{(6^2 - 5^2)}{6^5 - 5^5}$ (b) $\frac{(6^3 - 5^3)5^2}{6^5 - 5^5}$

(c) $\frac{(6^3 - 5^3)(6^2 - 5^2)}{6^5 - 5^5}$ (d) None of these

15. $P(B)$: The probability that player B and not A first obtain 6 is

(a) $\frac{(6^2 - 5^2)5^3}{6^5 - 5^5}$ (b) $\frac{(6^3 - 5^3)5^2}{6^5 - 5^5}$

(c) $\frac{(6^3 - 5^3)(6^2 - 5^2)}{6^5 - 5^5}$ (d) None of these

16. $P(C)$: The probability that players A and B obtain a 6 simultaneously is

(a) $\frac{(6^3 - 5^3)5^3}{6^5 - 5^5}$ (b) $\frac{(6^2 - 5^2)5^3}{6^5 - 5^5}$

(c) $\frac{(6^3 - 5^3)(6^2 - 5^2)}{6^5 - 5^5}$ (d) None of these

SECTION-IV

INTEGER ANSWER TYPE

This section contains 7 questions. The answer to each of the questions is a single–digit integer, ranging from 0 to 9. The bubble corresponding to the correct answer is to be darkened in the ORS. [4 marks for correct answer and 0 for wrong answer]

17. If the sum of squares of all integral values of a is S. For which the quadratic expression $(x - a)(x - 10) + 1$ can be factored as a product of two factors $(x + \alpha)(x + \beta)$ and $\alpha, \beta, \in I$, then $\frac{1}{26}S$ is equal to

18. $f(x)$ is a polynomial of 6^{th} degree and $f(x) = f(2 - x)$ $\forall \ x \in R$. If $f(x) = 0$ has 4 distinct real roots and two real and equal roots then sum of roots of $f(x) = 0$ is

19. If in a rectangular hyperbola $xy = c^2$, the locus of the middle points of chords of constant length $2d$ is $(x^2 + y^2)(xy - c^2) = d^2\lambda xy$, then the value of λ must be

20. Let the curves $a_1x^2 + 2h_1xy + b_1y^2 - 2g_1x - 2f_1y + c = 0$ and $a_2x^2 - 2h_2xy + (a_2 + a_1 - b_1)y^2 - 2g_2x - 2f_2y + c = 0$ $(a_1 \ne b_1)$ intersect at four concyclic points A, B, C and D. If H be the point $\left(\frac{g_1 + g_2}{a_1 + a_2}, \frac{f_1 + f_2}{a_1 + a_2}\right)$, then $\frac{5HA^2 + 7HB^2 - 4HC^2}{HD^2} =$

21. If $\phi^5 = 1$, $(\phi \ne 1)$, then $\frac{\phi}{1 + \phi^2} + \frac{\phi^2}{1 + \phi^4} + \frac{\phi^3}{1 + \phi} + \frac{\phi^4}{1 + \phi^3}$ equals

22. If number of ordered triplets (a, b, c) such that LCM$(a, b) = 1000$, LCM$(b, c) = 2000$ and LCM$(c, a) = 2000$ is α, then sum of the digits in α equals

23. Find the number of polynomials $f(x)$ with integral coefficient with $f(a) = b, f(b) = c, f(c) = a$ for distinct integers a, b, c.

ANSWERS

1. (a) **2.** (d) **3.** (c) **4.** (a) **5.** (d)
6. (c) **7.** (a) **8.** (a, c) **9.** (a, c, d)
10. (a, b, c, d) **11.** (a, b, c, d) **12.** (c)
13. (b) **14.** (b) **15.** (a) **16.** (c) **17.** (8)
18. (6) **19.** (2) **20.** (8) **21.** (2) **22.** (7)
23. (0)

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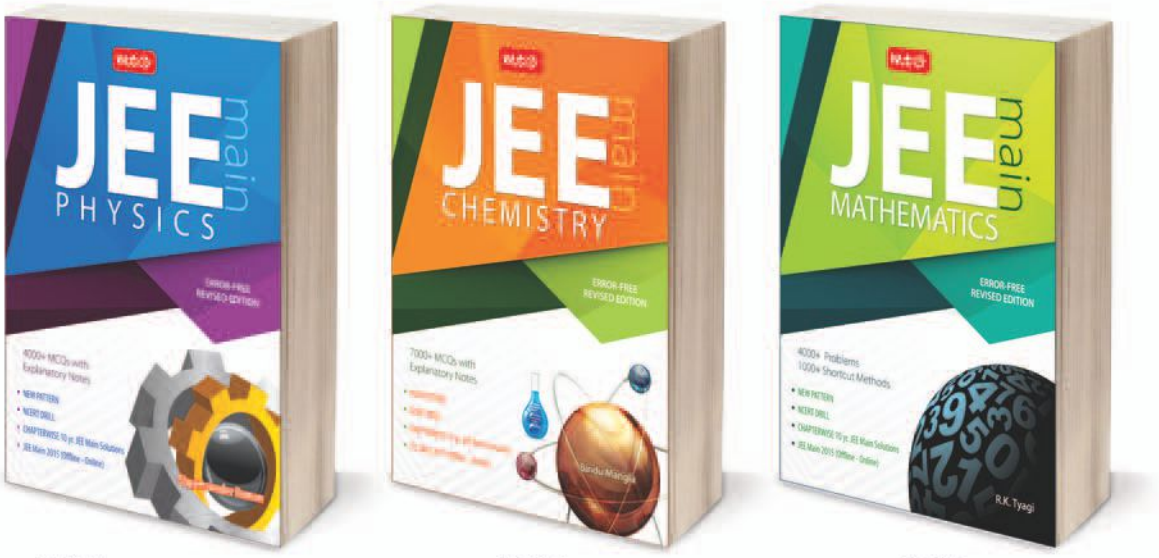

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Contd. from page no. 17

$$\begin{vmatrix} (1+\alpha)^2 & \alpha(2+3\alpha) & \alpha(2+5\alpha) \\ 3+2\alpha & 2\alpha & 2\alpha \\ 2 & 0 & 0 \end{vmatrix} = -648\alpha$$

Expanding, $2\alpha^2(3\alpha + 2) - 2\alpha^2(5\alpha + 2) = -324\alpha$

$\Rightarrow -4\alpha^3 = -324\alpha \Rightarrow \alpha(\alpha^2 - 81) = 0$

$\therefore \alpha = 0, -9, 9$

2nd solution

Writing the expression as a product of determinants, we have

$$\begin{vmatrix} 1 & \alpha & \alpha^2 \\ 4 & 2\alpha & \alpha^2 \\ 9 & 3\alpha & \alpha^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$$\Rightarrow \alpha^3 \begin{vmatrix} 1 & 1 & 1 \\ 4 & 2 & 1 \\ 9 & 3 & 1 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 1 & 4 & 9 \end{vmatrix} = -648\alpha$$

$\Rightarrow \alpha^3(-2)(4) = -648\alpha \Rightarrow \alpha^3 = 81\alpha$

$\therefore \alpha = 0, 9, -9$, as before

[Rating : Difficult]

11. (b, d) : The plane P_3 is given by

$(x + z - 1) + \lambda y = 0$ *i.e.* $x + \lambda y + z - 1 = 0$

Distance of the plane from (0, 1, 0) is

$$\left|\frac{\lambda - 1}{\sqrt{2+\lambda^2}}\right| = 1$$

$\Rightarrow \lambda^2 - 2\lambda + 1 = \lambda^2 + 2 \Rightarrow 2\lambda = -1 \therefore \lambda = -\frac{1}{2}$

Then the plane is $2x - y + 2z - 2 = 0$

The distance of (α, β, γ) from P_3 is 2

$$\Rightarrow \left|\frac{2\alpha - \beta + 2\gamma - 2}{\sqrt{9}}\right| = 2 \Rightarrow 2\alpha - \beta + 2\gamma - 2 = \pm 6$$

$\Rightarrow 2\alpha - \beta + 2\gamma - 8 = 0$ or $2\alpha - \beta + 2\gamma + 4 = 0$

[Rating : Easy]

12. (a, b) : By geometrical condition, line L is parallel to the line of intersection of P_1 and P_2.

A vector along L is

$(\hat{i} + 2\hat{j} - \hat{k}) \times (2\hat{i} - \hat{j} + \hat{k}) = \hat{i} - 3\hat{j} - 5\hat{k}$

Any point on L is $A(\lambda, -3\lambda, -5\lambda)$

The foot of perpendicular from A to plane P_1 is

$$\frac{\alpha - \lambda}{1} = \frac{\beta + 3\lambda}{2} = \frac{\gamma + 5\lambda}{-1} = -\frac{\lambda - 6\lambda + 5\lambda + 1}{1+4+1} = -\frac{1}{6}$$

$\therefore$ The foot of perpendicular is

$$\left(\lambda - \frac{1}{6}, -3\lambda - \frac{1}{3}, -5\lambda + \frac{1}{6}\right)$$

Only choice (a) and (b) match for some $\lambda \in R$.

[Rating : Difficult]

13. (a, d) : Let t_1 and t_2 be parameters of P and Q respectively, since OP and OQ are perpendicular

$\therefore$ Product of slopes $= -1$

$\Rightarrow t_1 t_2 = -4$

Again $[OPQ] = 3\sqrt{2}$

$$\Rightarrow \frac{1}{4}\left|t_1^2 t_2 - t_1 t_2^2\right| = 3\sqrt{2}$$

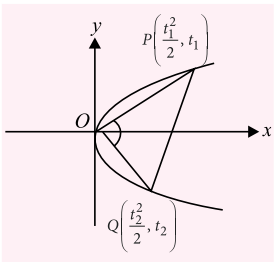
$$\Rightarrow |t_1 t_2||t_1 - t_2| = 12\sqrt{2}$$

$$\Rightarrow |t_1 - t_2| = 3\sqrt{2}$$

$$\Rightarrow \left|t_1 + \frac{4}{t_1}\right| = 3\sqrt{2} \quad \therefore t_1 + \frac{4}{t_1} = 3\sqrt{2}$$

$$\Rightarrow t_1^2 - 3\sqrt{2}t_1 + 4 = 0 \quad \therefore t_1 = 2\sqrt{2}, \sqrt{2}$$

Thus $P \equiv (4, 2\sqrt{2})$ or $(1, \sqrt{2})$



[Rating : Difficult]

14. (a, c) : $(1 + e^x)\frac{dy}{dx} + ye^x = 1$

$$\Rightarrow \frac{dy}{dx} + \frac{e^x}{1+e^x}y = \frac{1}{1+e^x}$$

This is linear *D.E.*, so

$$\text{I.F.} = e^{\int \frac{e^x}{1+e^x}dx} = e^{\ln(1+e^x)} = 1 + e^x$$

The solution is given by

$$y(1 + e^x) = \int \frac{1}{1+e^x}(1+e^x)dx + \lambda$$

$y(1 + e^x) = x + \lambda$

$(0, 2)$ is a point $\Rightarrow 2 \cdot 2 = \lambda \therefore \lambda = 4$

So, $y(1 + e^x) = x + 4$ ...(i)

(a) $x = -4 \Rightarrow y(1 + e^{-4}) = 0 \therefore y = 0$, (a) is true

(b) $x = -2 \Rightarrow y = \frac{2}{1+e^{-2}}$. So, (b) is not true.

(c) Differentiating (i) w.r.t. x, we get

$$\text{Also, } \frac{dy}{dx} = \frac{(e^x+1)\cdot 1 - (x+4)e^x}{(e^x+1)^2} = \frac{-e^x(x+3)+1}{(e^x+1)^2}$$

$$\text{Let } f(x) = \frac{-e^x(x+3)+1}{(e^x+1)^2}$$

$$f(0) = \frac{-2}{2^2} < 0, \quad f(-1) = \frac{\frac{-2}{e}+1}{\left(\frac{1}{e}+1\right)^2} > 0, \quad f(0)\cdot f(-1) < 0$$

Hence, $f(x)$ vanish at least once in $(-1, 0)$.

So, (c) is true.

[Rating : Difficult]

15. (b, c) : $(x - \alpha)^2 + (y - \alpha)^2 = r^2$

$\Rightarrow x^2 + y^2 - 2\alpha x - 2\alpha y + 2\alpha^2 - r^2 = 0$ ...(i)

Differentiating (i) w.r.t. x, we get

$2x + 2yy' - 2\alpha - 2\alpha y' = 0$ ...(ii)

$$\therefore \alpha = \frac{x + yy'}{1 + y'} \quad \text{...(iii)}$$

Differentiating (ii), we get

$$1 + y'^2 + yy'' - \alpha y'' = 0$$

$$\Rightarrow \quad 1 + y'^2 + yy'' - \frac{x + yy'}{(1 + y')} y'' = 0 \qquad \text{[From (iii)]}$$

Collecting terms, we get

$$(y - x)y'' + (1 + y' + y'^2)y' + 1 = 0$$

Now compare with given differential equation.

[Rating : Doable]

16. (a, d) : Rewrite f as

$$f(x) = \begin{cases} g(x), & x > 0 \\ 0 & x = 0 \\ -g(x), & x < 0 \end{cases}$$

We have, $f'(x) = \begin{cases} g'(x), & x > 0 \\ -g'(x), & x < 0 \end{cases}$

At $x = 0$

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{g(h) - 0}{h} = \lim_{h \to 0} \frac{g(h) - g(0)}{h} = g'(0)$$

$$f'(x) = \begin{cases} g'(x), & x \geq 0 \\ -g'(x), & x < 0 \end{cases}$$

f is differentiable at $x = 0$

$h(x) = e^{|x|}$

$\therefore$ h is not differentiable at $x = 0$

$$h'_+(0) = \lim_{t \to 0} \frac{h(0 + t) - h(0)}{t} = \lim_{t \to 0} \frac{e^t - 1}{t} = 1$$

$$h'_-(0) = \lim_{t \to 0} \frac{h(0 - t) - h(0)}{-t} = \lim_{t \to 0} \frac{e^t - 1}{-t} = -1$$

As $h'_+(0) \neq h'_-(0)$, h is not differentiable at origin.

hof is differentiable at 0, as f is differentiable at 0 and h is differentiable at $f(0)$.

But foh is not differentiable at 0.

[Rating : Difficult]

17. (a, b, c) : We have $f(x) = \sin\left(\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)\right)$

As $x \in R$

$$\sin\left(\frac{\pi}{2} \sin x\right) \in [-1, 1]$$

$$\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right) \in \left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$$

$$f(x) \in \left(-\frac{1}{2}, \frac{1}{2}\right)$$

$$fog(x) = \sin\left(\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin\left(\frac{\pi}{2} \sin x\right)\right)\right)$$

It is observed that as $\frac{\pi}{2} \sin x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, the range of $fog(x)$ is also $\left[-\frac{1}{2}, \frac{1}{2}\right]$

Again $\lim_{x \to 0} \frac{\sin\left(\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)\right)}{\frac{\pi}{2} \sin x}$

$$= \lim_{x \to 0} \left(\frac{\sin\left(\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)\right)}{\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)} \right) \cdot \frac{\frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)}{\frac{\pi}{2} \sin x}$$

$$= \lim_{x \to 0} \frac{\pi}{6} \frac{\sin\left(\frac{\pi}{2} \sin x\right)}{\frac{\pi}{2} \sin x} = \frac{\pi}{6}$$

Again $gof(x) = 1$

$$\Rightarrow \quad \frac{\pi}{2} \sin\left(\sin \frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)\right) = 1$$

$$\Rightarrow \quad \sin\left(\sin \frac{\pi}{6} \sin\left(\frac{\pi}{2} \sin x\right)\right) = \frac{2}{\pi} > \frac{1}{2}$$

So no x is possible.

[Rating : Difficult]

18. (a, c, d) : As $\vec{a} + \vec{b} + \vec{c} = 0$

$$\Rightarrow \quad a^2 = b^2 + c^2 + 2\vec{b} \cdot \vec{c}$$

$$\Rightarrow \quad 144 = 48 + c^2 + 48 \quad \Rightarrow \quad c^2 = 48 \Rightarrow c = 4\sqrt{3}$$

Again, $c^2 = a^2 + b^2 + 2\vec{a} \cdot \vec{b}$

$$\Rightarrow \quad \vec{a} \cdot \vec{b} = \frac{48 - 144 - 48}{2} = -72$$

$$|\vec{a} \times \vec{b} + \vec{c} \times \vec{a}| = |\vec{a} \times \vec{b} + \vec{a} \times \vec{b}| = 2|\vec{a} \times \vec{b}|$$

$$= 2\sqrt{a^2 b^2 - (\vec{a} \cdot \vec{b})^2} = 2\sqrt{12^2 \cdot 48 - (-72)^2}$$

$$= 2 \cdot 12\sqrt{48 - 36} = 2 \cdot 12 \cdot 2\sqrt{3} = 48\sqrt{3}$$

[Rating : Easy]

19. A → P, Q; B → P, Q; C → P, Q, S, T; D → Q, T

(A) $\left(\frac{\alpha\sqrt{3} + \beta}{2}\right) = \sqrt{3}$

Also, $\beta = \frac{\alpha - 2}{\sqrt{3}}$. So, $\alpha\sqrt{3} + \frac{\alpha - 2}{\sqrt{3}} = \pm 2\sqrt{3}$

$$\Rightarrow \quad \alpha\left(\frac{4}{\sqrt{3}}\right) = \frac{2}{\sqrt{3}} \pm 2\sqrt{3} = \frac{8}{\sqrt{3}}, \frac{-4}{\sqrt{3}}$$

$\therefore$ $\alpha = 2, -1 \Rightarrow |\alpha| = 1, 2$

(B) Continuity gives $b + a^2 = -2 - 3a$

Differentiability gives $b = -6a$

Eliminating 'b', we get

$a^2 - 3a + 2 = 0 \quad \therefore a = 1, 2$

(C) $(3 - 3\omega + 2\omega^2)^{4n+3} + (2 + 3\omega - 3\omega^2)^{4n+3} + (-3 + 2\omega + 3\omega^2)^{4n+3} = 0$

$\Rightarrow (3 - 3\omega + 2\omega^2)^{4n+3}\{1 + \omega^{4n+3} + (\omega^2)^{4n+3}\} = 0$
$1 + \omega^{\lambda} + \omega^{2\lambda} = 0$ as λ is not a multiple of 3
i.e. $4n + 3 \neq 3k$
i.e. $4n \neq 3(k - 1)$
So n is not a multiple of 3.

(D) By given condition $\frac{2ab}{a+b} = 4$ ...(i)
Also, $b - 5 = 2(5 - a)$
$\therefore \quad b = 15 - 2a$
On putting b in (i), we get
$2a(15 - 2a) = 4(15 - a)$
$\Rightarrow 2a^2 - 17a + 30 = 0 \Rightarrow (2a - 5)(a - 6) = 0$
$\therefore \quad a = \frac{5}{2}, 6$
$|q - 2a| = |10 - 2a| = |10 - 5| = 5$
or $= |10 - 12| = 2$

[Rating : Easy]

20. A → P, R, S; B → P; C → P, Q; D → S, T

(A) As $2(a^2 - b^2) = c^2$
$\Rightarrow 2(\sin^2 X - \sin^2 Y) = \sin^2 Z$
$\Rightarrow 2 \sin(X + Y)\sin(X - Y) = \sin^2 Z$
$\therefore \quad \sin|X - Y| = \frac{\sin Z}{2}$
$\cos(n\pi\lambda) = 0 \Rightarrow \quad \cos\left(\frac{n\pi}{2}\right) = 0$
$\Rightarrow \quad \frac{n\pi}{2} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$
$\therefore \quad n = 1, 3, 5$

(B) $1 + \cos 2X - 2\cos 2Y = 2 \sin X \sin Y$
$\Rightarrow 2\cos^2 X - 2\{1 - 2\sin^2 Y\} = 2 \sin X \sin Y$
$\Rightarrow \sin^2 X + \sin X \sin Y - 2\sin^2 Y = 0$
$\Rightarrow a^2 + ab - 2b^2 = 0$
$\Rightarrow \left(\frac{a}{b}\right)^2 + \left(\frac{a}{b}\right) - 2 = 0 \quad \Rightarrow \quad \left(\frac{a}{b} + 2\right)\left(\frac{a}{b} - 1\right) = 0$
$\therefore \quad \frac{a}{b} = -2, 1 \qquad \therefore \quad \frac{a}{b} = 1$

(C) Angle bisector of OX and OY is $y = x$
Now distance of $(\beta, 1 - \beta)$ from $x - y = 0$ is
$\left|\frac{\beta - (1 - \beta)}{\sqrt{2}}\right| = \frac{3}{\sqrt{2}}$
$\Rightarrow |2\beta - 1| = 3 \quad \Rightarrow 2\beta - 1 = \pm 3 \quad \therefore \quad \beta = 2$ or -1
$\therefore \quad |\beta| = 2, 1$

(D) For $\alpha = 0$, $y = 3$
For $\alpha = 1$
$y = |x - 1| + |x - 2| + x$
$$= \begin{cases} 3 - x, & x < 1 \\ 1 + x, & 1 \le x < 2 \\ 3x - 3, & x \ge 2 \end{cases}$$
$\text{Area} = \frac{1}{2}(2 + 3)\cdot 1 + \frac{1}{2}(2 + 3) - \int_0^2 2\sqrt{x}\,dx \quad = 5 - \frac{8\sqrt{2}}{3}$
$\therefore \quad F(1) + \frac{8}{3}\sqrt{2} = 5$

For $\alpha = 0$, $y = 3$
$\therefore \quad \text{Area} = 6 - \int_0^2 2\sqrt{x}\,dx \quad = 6 - \frac{8}{3}\sqrt{2}$
$\therefore \quad F(0) + \frac{8}{3}\sqrt{2} = 6$

[Rating : Difficult]

PAPER-2

1. (4) : 1st solution :

$$\alpha_k = \cos\frac{2k\pi}{14} + i\sin\frac{2k\pi}{14} = e^{i\frac{2\pi k}{14}}$$

$$\frac{\sum_{k=1}^{12}\left|e^{i2(k+1)\frac{\pi}{14}} - e^{i\frac{2k\pi}{14}}\right|}{\sum_{k=1}^{3}\left|e^{\frac{i2(4k-1)\pi}{14}} - e^{i\frac{2(4k-2)\pi}{14}}\right|} = \frac{\sum_{k=1}^{12}\left|e^{i\frac{2\pi}{14}} - 1\right|}{\sum_{k=1}^{3}\left|e^{i\frac{2\pi}{14}} - 1\right|} = \frac{12}{3} = 4$$

2nd solution : The expression has a geometric meaning. $|\alpha_{k+1} - \alpha_k|$ is the side length of a polygon having 14 sides as α_k's are 14^{th} roots of unity.
The expression $= \frac{12}{3} = 4$
Recall, $\alpha_k = \cos\frac{k\pi}{7} + i\sin\frac{k\pi}{7} = \cos\frac{2\pi k}{14} + i\sin\frac{2k\pi}{14}$

[Rating : Difficult]

2. (9) : $\frac{S_7}{S_{11}} = \frac{6}{11} \Rightarrow \frac{7(2a + 6d)}{11(2a + 10d)} = \frac{6}{11}$
$\Rightarrow 7a + 21d = 6a + 30d \Rightarrow \quad a = 9d$
Now, $130 < a + 6d < 140$
i.e. $130 < 15d < 140$
Hence $d = 9$

[Rating : Easy]

3. (8) : x^9 can be obtained as a sum of two numbers in 5 ways, *i.e.*, (0, 9), (1, 8), (2, 7), (3, 6), (4, 5) and as a sum of three numbers in 3 groups, *i.e.*, (1, 2, 6), (1, 3, 5), (2, 3, 4).
Thus the total number of ways = 8

[Rating : Doable]

4. (4) : $e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{5}{9} = \frac{4}{9}$

The foci are $(\pm ae, 0)$ *i.e.* (2, 0) and (−2, 0).
The parabola P_1 is $y^2 = 8x$ and P_2 is $y^2 = -16x$

As tangent with slope m_1 to P_1 passes through (–4, 0), we have

$$y = m_1x + \frac{2}{m_1} \text{ giving } 0 = -4m_1 + \frac{2}{m_1}$$

i.e. $4m_1^2 = 2 \Rightarrow m_1^2 = \frac{1}{2}$

Again for tangent with slope m_2 to P_2 passing through (2, 0), we have

$$y = m_2x - \frac{4}{m_2} \Rightarrow 0 = 2m_2 - \frac{4}{m_2}$$

$\Rightarrow \quad 2m_2^2 = 4 \quad \therefore \; m_2^2 = 2$

Thus, $\frac{1}{m_1^2} + m_2^2 = 2 + 2 = 4$

[Rating : Doable]

5. (2) : $\lim\limits_{\alpha\to 0}\left(\frac{e^{\cos(\alpha^n)} - e}{\alpha^m}\right) = \lim\limits_{\alpha\to 0}\frac{e(e^{\cos(\alpha^n)-1} - 1)}{\alpha^m}$

$$= \lim_{\alpha\to 0}\frac{e(\alpha^n)^2}{\alpha^m}\left\{\frac{e^{\cos(\alpha^n)-1}-1}{\cos(\alpha^n)-1}\right\}.\left\{\frac{\cos(\alpha^n)-1}{(\alpha^n)^2}\right\} = \frac{-1}{2}e\alpha^{2n-m}$$

As limit is given to be $-e/2$, we have

$2n - m = 0. \quad \therefore \; m = 2n$

[Rating : Easy]

6. (9) : $\alpha = \int\limits_0^1 e^{(9x+3\tan^{-1}x)}\left(\frac{12+9x^2}{1+x^2}\right)dx$

Set $9x + 3\tan^{-1}x = t$

so that $\frac{dt}{dx} = 9 + \frac{3}{1+x^2} = \frac{12+9x^2}{1+x^2}$

We have, $\alpha = \int\limits_0^{9+\frac{3\pi}{4}} e^t dt = e^{9+\frac{3\pi}{4}} - 1$

$\therefore \; \ln|\alpha+1| = 9 + \frac{3\pi}{4}$

Thus $\ln|\alpha+1| - \frac{3\pi}{4} = 9$

[Rating : Easy]

7. (7) : $\lim\limits_{x\to 1}\frac{F(x)}{G(x)} = \lim\limits_{x\to 1}\frac{F'(x)}{G'(x)} = \lim\limits_{x\to 1}\frac{f(x)}{x\,|f(f(x))|} = \frac{1}{14}$

$$= \frac{f(1)}{\left|f\left(\frac{1}{2}\right)\right|} = \frac{1}{14}$$

Thus $\left|f\left(\frac{1}{2}\right)\right| = 7$. But $f\left(\frac{1}{2}\right) \neq -7$ as $f(x)$ vanishes exactly at one point.

[Rating : Difficult]

8. (9) : $\vec{s} = 4\vec{p} + 3\vec{q} + 5\vec{r}$

Let $-\vec{p} + \vec{q} + \vec{r} = \vec{\alpha}$, $\vec{p} - \vec{q} + \vec{r} = \vec{\beta}$ and $-\vec{p} - \vec{q} + \vec{r} = \vec{\gamma}$

Then solving for $\vec{p}$, $\vec{q}$ and $\vec{r}$, we have

$$\vec{p} = \frac{\vec{\beta} - \vec{\gamma}}{2}, \; \vec{q} = \frac{\vec{\alpha} - \vec{\gamma}}{2}, \; \vec{r} = \frac{\vec{\alpha} + \vec{\beta}}{2}$$

Rewrite, $\vec{s} = 4\left(\frac{\vec{\beta} - \vec{\gamma}}{2}\right) + 3\left(\frac{\vec{\alpha} - \vec{\gamma}}{2}\right) + 5\left(\frac{\vec{\alpha} + \vec{\beta}}{2}\right)$

$$= 4\vec{\alpha} + \frac{9}{2}\vec{\beta} - \frac{7}{2}\vec{\gamma}$$

$\therefore \; x = 4, \; y = \frac{9}{2}, \; z = \frac{-7}{2}$

Thus, $2x + y + z = 9$

[Rating : Difficult]

9. (a, d) : We have $\alpha x^2 - x + \alpha = 0$

$D = 1 - 4\alpha^2$

For distinct real roots, $1 - 4\alpha^2 > 0$

i.e., $\alpha \in \left(-\frac{1}{2}, \frac{1}{2}\right)$

Now, $|x_1 - x_2| < 1$ *i.e.* $(x_1 - x_2)^2 < 1$

i.e., $\frac{b^2 - 4ac}{a^2} < 1 \Rightarrow 1 - 4\alpha^2 < \alpha^2$

$$\Rightarrow \; \alpha \in \left(-\infty, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \infty\right)$$

Combining the two bounds, we have

$$\alpha \in \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$$

$$\therefore \; S = \left(-\frac{1}{2}, -\frac{1}{\sqrt{5}}\right) \cup \left(\frac{1}{\sqrt{5}}, \frac{1}{2}\right)$$

[Rating : Doable]

10. (b, c, d) : $\alpha = 3\sin^{-1}\frac{6}{11} > 3\sin^{-1}\frac{6}{12} = \frac{3\pi}{6} = \frac{\pi}{2}$

$$\Rightarrow \; \alpha > \frac{\pi}{2}$$

$\beta = 3\cos^{-1}\frac{4}{9} > 3\cos^{-1}\frac{4}{8}$ *i.e.*, $\beta > \pi$

Then, $\alpha + \beta > \frac{3\pi}{2}$

Also, $\alpha + \beta$ lies in fourth quadrant, hence $\cos(\alpha + \beta) > 0$

[Rating : Difficult]

11. (a, b) : The equation to the ellipses are

$$E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \; (a > b) \text{ and } E_2 : \frac{x^2}{c^2} + \frac{y^2}{d^2} = 1 \; (c < d)$$

The tangent is $x + y = 3$

The tangent at (α, β) to $x^2 + (y-1)^2 = 2$ is

$x\alpha + (y-1)(\beta-1) = 2$

i.e. $x\alpha + (\beta-1)y = \beta + 1$

Giving $\alpha = 1, \beta = 2$

Hence P is $(1, 2)$.

$x + y = 3$ touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at (x_1, y_1) giving

$$\frac{xx_1}{a} + \frac{yy_1}{b} = 1$$

Comparing, we get $Q\left(\frac{a}{3}, \frac{b}{3}\right)$

Again it touches $\frac{x^2}{c^2} + \frac{y^2}{d^2} = 1$, we get $R\left(\frac{c}{3}, \frac{d}{3}\right)$

$$\left(\frac{a}{3}-1\right)^2 + \left(\frac{b}{3}-2\right)^2 = \frac{8}{9}$$

Also, $a + b = 9$, we get $(a, b) = (5, 4)$

Similarly, $(c, d) = (1, 8)$

$$e_1^2 = 1 - \frac{4}{5} = \frac{1}{5} \; ; \; e_2^2 = 1 - \frac{1}{8} = \frac{7}{8}$$

[Rating : Difficult]

12. (a, b, d) : Let $1 + x^2 - y^2 = 1$

The family of circles touching hyperbola at (x_1, y_1) is

$(x - x_1)^2 + (y - y_1)^2 + \lambda(xx_1 - yy_1 - 1) = 0$

As centre is $(x_2, 0)$, we have $2y_1 + \lambda y_1 = 0$

$\therefore \quad \lambda = -2$

Also, $x_2 = 2x_1$

Then $P \equiv (x_1, y_1) = (x_1, \sqrt{x_1^2 - 1}) \qquad (\because x_1^2 - y_1^2 = 1)$

$N \equiv (2x_1, 0)$

Again, $M \equiv \left(\frac{1}{x_1}, 0\right)$

Here, $l = x_1 + \frac{1}{3x_1} \quad \therefore \quad \frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}, \; x_1 > 1$

And, $\frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$ as $m = \sqrt{\frac{x_1^2 - 1}{3}}$

As $m = \frac{y_1}{3}; \; \frac{dm}{dy_1} = \frac{1}{3}$

[Rating : Difficult]

13. (a, c) : $I_1 = \int_0^{4\pi} e^t (\sin^6 at + \cos^4 at)dt$

$$= \sum_{k=0}^{3} \int_{k\pi}^{(k+1)\pi} e^t (\sin^6 at + \cos^4 at)dt$$

$$= \sum_{k=0}^{3} \int_0^{\pi} e^{(k\pi + t)} (\sin^6 at + \cos^4 at)dt$$

$$= \left(\sum_{k=0}^{3} e^{k\pi}\right) \int_0^{\pi} e^t (\sin^6 at + \cos^4 at)dt$$

$$= (1 + e^{\pi} + e^{2\pi} + e^{3\pi}) \int_0^{\pi} e^t (\sin^6 at + \cos^4 at)dt$$

$\therefore$ The given expression

$$= 1 + e^{\pi} + e^{2\pi} + e^{3\pi} = \frac{e^{4\pi} - 1}{e^{\pi} - 1}$$

Also, $a = 2$ and $a = 4$ both cases hold.

[Rating : Difficult]

14. (b, c) : Define $H(x) = f(x) - 3g(x)$

$H(-1) = 3, \; H(0) = 3, \; H(2) = 3$

Then $H'(x)$ will vanish at least twice in $(-1, 0) \cup (0, 2)$

As $H''(x)$ has no root in $(-1, 0)$ and $(0, 2)$, the equation $H'(x) = 0$ has exactly one root in $(-1, 0)$ and exactly one root in $(0, 2)$.

[Rating : Difficult]

15. (a, b) : $f(x) = (7\tan x^6 - 3\tan^2 x)(\tan^2 x + 1)$

$= (7\tan^6 x - 3\tan^2 x)\sec^2 x$

Now, $\int_0^{\pi/4} f(x)dx = \int_0^1 (7t^6 - 3t^2)dt = \left(t^7 - t^3\right)\Big|_0^1 = 0$

Again, $\int_0^{\pi/4} x\, f(x)dx = \int_0^1 \tan^{-1} t(7t^6 - 3t^2)dt$

$$= \tan^{-1} t[t^7 - t^3]_0^1 - \int_0^1 (t^7 - t^3)\frac{1}{1+t^2}dt$$

$$= \int_0^1 t^3(1 - t^2)dt = \frac{t^4}{4} - \frac{t^6}{6}\Big|_0^1 = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$$

[Rating : Difficult]

16. (d) : We have, $f'(x) = \frac{192x^3}{2 + \sin^4 \pi x}$

Given, $f\left(\frac{1}{2}\right) = 0$ and $m \le \int_{1/2}^{1} f(x)dx \le M$

$\therefore$ $f(x)$ is increasing in $\left(\frac{1}{2}, 1\right)$.

$\therefore \quad f'(x)_{\max} = \frac{192}{24} = 96$

$\Rightarrow \quad 96 = \frac{f(1) - f(1/2)}{1/2} \quad \Rightarrow \quad f(1) = 96 \times \frac{1}{2} = 48$

$$M = \frac{1}{2} \times \frac{1}{2} \times 48 = 12$$

$$f'(x)_{\text{min.}} = \frac{\left(\frac{192}{8}\right)}{3} = \frac{192}{24}$$

$$\Rightarrow \quad \frac{192}{24}=\frac{f(1)-0}{(1/2)} \quad \Rightarrow \quad f(1)=\frac{192}{24}\times\frac{1}{2}$$

$$\therefore \quad m=\frac{1}{2}\times\frac{1}{2}\times\frac{192}{24}\times\frac{1}{2}=1$$

[Rating : Difficult]

17. (a, b) : **Box I :** red balls $\rightarrow n_1$
black balls $\rightarrow n_2$
Box II : red balls $\rightarrow n_3$
black balls $\rightarrow n_4$

$$P(R)=\frac{1}{2}\cdot\frac{n_1}{n_1+n_2}+\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}$$

$$P(\text{Box II}/R)=\frac{\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}}{\frac{1}{2}\cdot\frac{n_1}{n_1+n_2}+\frac{1}{2}\cdot\frac{n_3}{n_3+n_4}}$$

$$=\frac{1}{1+\left(\frac{\frac{n_1}{n_1+n_2}}{\frac{n_3}{n_3+n_4}}\right)}=\frac{1}{3}$$

Thus, $\frac{n_1}{n_1+n_2}=2\frac{n_3}{n_3+n_4}$

i.e. $2\left(1+\frac{n_2}{n_1}\right)=1+\frac{n_4}{n_3}$ *i.e.* $\frac{n_4}{n_3}-2\frac{n_2}{n_1}=1$

[Rating : Doable]

18. (c, d) : From the given condition

$$\frac{n_1}{n_1+n_2}\cdot\frac{n_1-1}{n_1+n_2-1}+\frac{n_2}{n_1+n_2}\cdot\frac{n_1}{n_1+n_2-1}=\frac{1}{3}$$

i.e. $3n_1(n_1+n_2-1)=(n_1+n_2)^2-n_1-n_2$
$\Rightarrow \quad 3n_1(n_1+n_2-1)=(n_1+n_2)(n_1+n_2-1)$
$\Rightarrow \quad 3n_1=n_1+n_2 \quad \therefore \quad 2n_1=n_2$

[Rating : Doable]

19. (a, b, c) : $f'(x)=xF'(x)+F(x)$
$\therefore \quad f'(1)=F(1)+F'(1)<0$
$f(2)=2F(2)<0$
Thus $f'(x)=xF'(x)+F(x)<0$

[Rating : Difficult]

20. (c, d) : $\int_1^3 x^2F'(x)dx=-12$

As $\int_1^3 x^3F''(x)dx=40$, we have

$$x^3F'(x)\Big|_1^3-3\int_1^3 x^2F'(x)dx=40$$

$$\Rightarrow \quad x^2(f'(x)-F(x)\big|_1^3-3(-12)=40$$
$\Rightarrow \quad 9(f'(3)-F(3))-(f'(1)-F(1))=4$
$\Rightarrow \quad 9f'(3)+36-f'(1)=4$
Thus, $9f'(3)-f'(1)+32=0$
Again, $\int_1^3 x^2F'(x)dx=-12$

i.e. $x^2F(x)\big|_1^3-\int_1^3 2xF(x)dx=-12$

i.e. $-36-2\int_1^3 f(x)dx=-12$

$$\therefore \quad \int_1^3 f(x)dx=-12$$

[Rating : Difficult]

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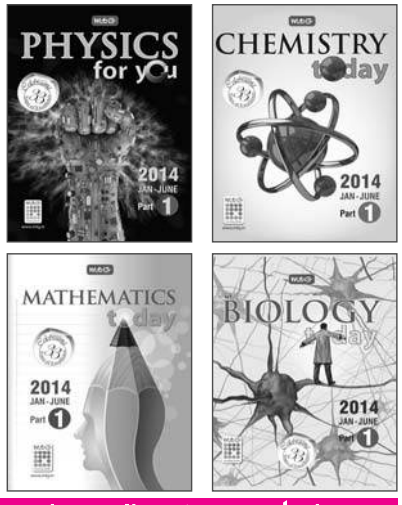

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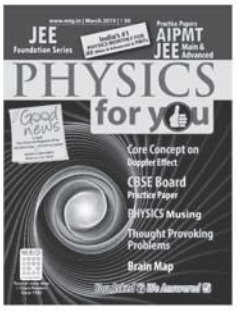

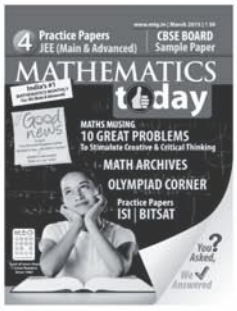
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